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ABSTRACT

This monograph was developed with the intention of addressing the concerns of high school mathematics teachers in Alberta (Canada) who want to base their programs on problem solving but have questions about effective and efficient ways to do so. Considered are the most basic philosophical questions, and a framework is provided to use in solving problems. Contained is a section which includes ideas for teachers to use in their own classrooms as they encourage their students to become better problem solvers. In addition, sample problems are presented, along with their prerequisites, complete solutions, and some suggestions for extensions. A problem bank organized by strands is presented from which teachers can obtain some additional problems for use in their classroom. A two-part bibliography of six pages contains suggestions on where to obtain information on the pedagogy of problem solving, as well as where to obtain more problems. (TW)

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ED295799

Senior High School

MONOGRAPH

PROBLEM SOLVING IN MATHEMATICS: FOCUS FOR THE FUTURE

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PROBLEM SOLVING IN MATHEMATICS: FOCUS FOR THE FUTURE

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Throughout this monograph, the SI symbol 'a' has been used to represent years.

PREFACE

The beliefs about mathematics and the attitudes toward it that are held by students, teachers, and members of the general public have great diversity. At one extreme there is "math anxiety", strong dislike for mathematics, poor performance, and avoidance of it. At the other extreme there is enjoyment and success with mathematics and an active pursuit of opportunities to use mathematics in solving problems that arise from daily life or in further formal study.

Those who dislike and avoid mathematics often view the subject as the study of rules, routines, and techniques that are to be learned by heart from a teacher who is the unquestioned authority. Persons who enjoy mathematics recognize the importance of standard mathematical procedures, but they also view the doing of mathematics as a personal, creative act that is understandable and logical, often visual and intuitive, and usually marked by more than one way of coming to understand, explain, and solve problems. There can be no doubt that facility with routines (e.g., for computation) has great value, but this value is only fully realized when the person doing mathematics is able to invent a mathematical procedure or recall and apply a previously learned one in order to solve a problem that has not been encountered before in exactly its present form.

Although most mathematics teachers want their students to achieve success and to have positive attitudes toward mathematics, many mathematics students seem to view mathematics in a way that is negative and counter-productive. For example, evidence from the National Assessment of Educational Progress in the United States (Carpenter, et al., 1981) and from the British project Concepts in Secondary Mathematics and Science (Hart et al., 1981) suggests that many students assume a passive role in their mathematics classes, have a very superficial understanding of the mathematics they study, and encounter great difficulty when they are confronted with problems that are non-routine.

The recent movement to focus the mathematics curriculum on problem solving rather than merely on computational skill takes these facts into account. By presenting core topics not just as sets of terms, formulas, and propositions to be learned, but as flexible and adaptable thinking tools that can be developed to tackle new problems, teachers hope their students will develop a new and more positive outlook. Teachers are also challenged to act as good role models as they present an explicit model of problem solving processes and demonstrate for students the use of appropriate problem solving strategies.

In the curriculum change process, both students and teachers are faced with new roles and opportunities to change. They may become actively involved with mathematics clubs and contests; they may try out organizational patterns such as cooperative learning and peer tutoring; they may work on communication skills such as comprehending, organizing, convincing, and explaining. This book, which addresses these and other issues, is designed to be a practical guide for teachers (and prospective teachers) as they seek to make problem solving a part of the daily experience in their classrooms.

INTRODUCTION

There should be little doubt in any mathematics educator's mind about the importance of problem solving. It was identified by the National Council of Teachers of Mathematics as the "focus" of school mathematics for the 1980s. In the *Secondary Education in Alberta* Policy Statement of 1985, it was listed as one of the primary goals of secondary education. The curriculum revisions which will shape mathematics education in Alberta over the next few years will place an increasing emphasis on problem solving. With these facts in mind, problem solving must become an integral part of every mathematics classroom.

This monograph was created to address some of the concerns of high school teachers who want to base their programs on problem solving but have questions about the most effective and efficient ways to do so. It discusses the most basic philosophical questions and provides a framework to use in solving problems. It provides a section which will give teachers ideas to use in their own classrooms as they encourage their own students to become better problem solvers. In addition, reading and writing and their importance to effective problem solving is discussed using the strands of the Language Arts/English program as guides.

Many teachers have shown a willingness to incorporate problem solving into their classrooms but have been concerned that they will be forced to take time away from the presentation of required material. How to use problem solving techniques throughout the presentation of core topics is examined in this monograph.

Teachers have expressed concerns about how to evaluate problem solving properly. Questions have arisen with regard to how solutions should be marked, evaluated, tested and reported. The monograph addresses some of these concerns.

A major use of a document such as this is as a source of problems. In the section on project problems, a "rich" problem and its in-depth solution is presented. Several other problems are presented with their prerequisites, complete solutions and some suggestions for extension. A problem bank organized by strands is presented from which teachers can obtain some good problems for use in their classrooms. Finally, a two part bibliography rounds out the monograph with some suggestions on where to obtain information on the pedagogy of problem solving as well as where to obtain more problems.

Problem solving is something which is learned by doing. Teachers should feel comfortable about using this document, the ideas and the problems, even if they are just beginning to see problem solving as an integral part of their mathematics classrooms. Students will become good problem solvers only if they think that the activity is important - and they will believe that it is important if they see their teachers solving problems and expecting them to do the same.

THE NATURE OF PROBLEM SOLVING

WHAT IS A PROBLEM?

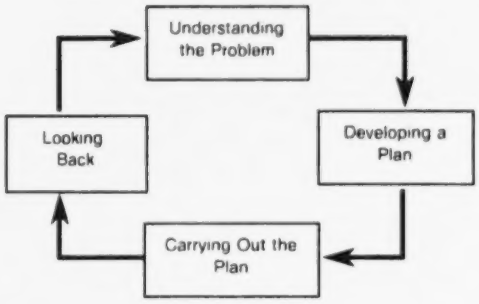
Mathematics and mathematics educators agree on the importance of problem solving. They do not however, agree on precisely what a problem is. For example, Skinner (1966) suggested that it is "A question for which there is at the moment no answer" and Newell and Simon (1972) stated that "A person is confronted with a problem when he wants something and does not know immediately what series of actions he can perform to get it." Others have suggested that problems and problem solving are not exclusively the domain of human beings. For example, Duncker (1945) contended that a problem is "when a living creature has a goal but does not know how this goal is to be reached" and Davis (1973) proposed that a problem is "a stimulus situation for which an organism does not have a ready response." Radford and Burton (1974) were more forthright when they suggested that a problem is "any situation in which the end result cannot be reached immediately."

George Polya, in an article in the 1980 NCTM Yearbook, said "Solving a problem is finding the unknown means to a distinctly conceived end. If the end by its simple presence does not instantaneously suggest the means, if, therefore, we have to search for the means, reflecting consciously how to attain the end, we have to solve a problem. To solve a problem is to find a way where no way is known off-hand, to find a way out of a difficulty, to find a way around an obstacle, to attain a desired end, that is not immediately attainable, by appropriate means". Polya added that "Solving problems is the specific achievement of intelligence, and intelligence is the specific gift of man" and concluded that "Solving problems is human nature itself."

A PROBLEM SOLVING MODEL

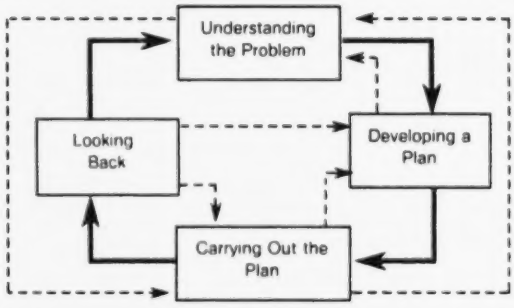
Many problems can be solved with the use of more than one strategy. Because it is often not immediately clear which approach to use, students find it useful to have a framework within which to work.

In 1945, George Polya published How to Solve It in which he outlined a four-step model which could be used during the solution of problems.



Problem solvers can learn individual skills and strategies which can be used within this framework, and as they broaden their knowledge their facility for solving problems will improve.

Polya's framework is not fixed. Although problem solvers may approach a solution in the order outlined, they will often return to earlier stages because they have encountered an obstacle or because it becomes obvious that another approach will work better.



A PROBLEM SOLVING FRAMEWORK

Understanding the Problem

In this stage of the model, the problem solver should begin to think about the problem and attempt to clarify what is given and what is being asked. In order to understand the problem, the solver may have to

- read the problem several times

- refer to other resources to clarify the meaning of key words and phrases
- restate the problem using familiar terms
- evaluate the data in the problem to determine whether there is sufficient information to solve the problem and whether any of the data is extraneous
- determine if there are hidden assumptions which contain information necessary to the solution of the problem
- diagram, simulate or model the problem situation
- consider alternative interpretations.

Developing a Plan

Once a problem is understood, a plan for solution should be developed. Since problems may be solved in a variety of ways, a problem solver should give thought to the different strategies which may be employed and try to choose the one (or the combination) which seems most appropriate. However, a good problem solver should be prepared to revise a plan if an initial strategy proves unworkable.

Some of the strategies that may be employed in solving problems are to

- work backwards
- simulate, model or diagram the situations in the problem
- use organized lists, charts, tables or graphs
- look for a pattern
- select appropriate notation
- identify given, needed and wanted information
- use a formula or write an equation
- solve a simpler, related problem
- partition a problem and solve each part
- use logical deduction
- check for hidden assumptions
- guess an answer, test it and adjust the guess
- change the point of view.

Carrying Out the Plan

After a plan has been developed, the next stage is to implement the planned strategy and attempt to solve the problem. In carrying out the plan, it is essential that problem solvers continually monitor their progress in order to determine whether or not their plan will lead to a solution. The solver may, in fact, find that the plan will not produce a solution, in which case a new plan will have to be developed.

In monitoring progress toward a solution to a problem, the solver should ask

- Is my plan leading toward a solution?
- Do I need to alter my plan, or perhaps abandon it for a new one?
- Do I need additional information or assistance in carrying out my plan?
- Am I carefully documenting my effort in order to keep track of my progress?

Looking Back

After the plan has been worked through and the solution has been achieved, the problem solver should look back and reflect on the process. It is important that the solution answers what the problem originally asked and that it makes sense within the context of the problem.

The looking back stage is a time to extend what has been accomplished. In this stage, the problem solver should internalize the successful strategies and processes for future use as well as look for different approaches to a solution. A good problem solver will extend the problem beyond its stated parameters; perhaps to answer unasked questions, generalize a solution or examine a related situation.

The problem should be well documented so that its solution will be clear to others, and the problem solver should be able to communicate and defend both processes and results.

A problem solver should answer the following questions during the looking back phase:

- Is my answer reasonable?
- What have I learned that I didn't know before?
- What are the features of this problem that I will recognize in other problems?
- Can I make and solve related problems?
- Can I explain, document or generalize the solution?
- Have I found all of the solutions?
- Have I tried strategies other than the one which first gave me the solution?

TEACHING TECHNIQUES AND TACTICS

ORGANIZING FOR PROBLEM SOLVING

Cooperative Learning

Cooperative learning is a learning concept receiving more and more attention each year. The essence of cooperative learning is to place students into groups in which everyone is responsible for the learning of each person in the group. The achievement of each individual then becomes a cooperative effort. It is not necessary to use the entire process in every class; rather, it is a package from which teachers may choose according to the needs of their classes.

Different teachers use a variety of methods in setting up groups depending on the ages of the students, the concepts being taught and the available resources. There should be the same cross-section of individual achievement within each group. Every group should include a complete range of student achievement from high achievers through to average and low achieving students.

The group may not seek assistance from the teacher unless everyone in the group has a hand up seeking help for the same question. If someone in the group knows the solution to the group's problem, it is the responsibility of that person to teach the rest of the group. This approach is based on the idea *that learning is enhanced when we prepare to teach someone else.*

Not all the teaching is done on a cooperative basis, it is merely another tool in the teaching/ learning process. If the teacher is interested in developing basic concepts, these may be taught to the entire class. The application and use of the basic concepts can be handled better in a cooperative learning situation.

Assignments are set up so that members of the group are individually responsible for a concept that they must teach to the others in their group. For example, in a trigonometry lesson on transformations of trigonometric graphs, the teacher could give an assignment to each student in the group to be responsible for one type of transformation. Since amplitude and vertical translation are easier concepts to deal with, the slower students might receive these topics to develop and teach to the rest of the group. The better students could be given the change of period and phase shift to develop and teach. The assignment for the group could be a question involving all four transformations. This process of differentiated responsibilities is known as "jigsawing". Assignments are usually given on a timed basis or as homework. This keeps the students on task while working on the assignment in class or if given as homework, forces them to contribute to the group effort the following day.

Each student in the group receives a base-mark at the beginning of the year based on the mark of the previous year and individual tests and quizzes in the first few weeks of class. This base-mark is changed each month, being updated as more tests and assignments are given. On any individual assignment, when a student receives a mark higher than his or her base-mark, the whole group will receive one bonus mark. A group of five students could receive five bonus marks if all were above their base-mark. If in a group of five students, three were below their base-mark and two were above, the bonus marks would total -1. There would be no deductions however, and the marks would be as originally received. This way all students in the group must strive to be above their base-mark if they wish to receive the five bonus marks. Otherwise the good students are always carrying the weaker students. Marks are modified should anyone receive a mark above one hundred percent. This method of marking provides the incentive for the group members to help everyone else in their group so that more bonus marks can be achieved.

Having students working together gets the students answering questions for one another that would normally be directed toward the teacher. This process means that students must be able to speak the language of mathematics, something which is poorly done most of the time. Students must use the correct vocabulary, listen to others, contribute to the group, defend their ideas and challenge the ideas of others. The teacher only has to

monitor six or seven groups rather than thirty-five individual students. Solutions to problems are presented to the entire class and the teacher selects anyone in the group to give the presentation. This ensures that every individual understands the process by which a solution is achieved.

An excellent introduction for teachers wishing information on cooperative learning is Dishon, Dee and Pat Wilson O'Leary. A Guidebook for Cooperative Learning: A Technique for Creating More Effective Schools. Holmes Beach, FL: Learning Publications, 1984.

Notebooks for Problem Solving

Students should be encouraged to keep an accurate account of all trials or attempts at solving problems. They should be rewarded for their progress through a problem - not just for right answers.

A coil bound notebook may be used or a student workbook produced, in which the students should be encouraged to enter:

- a statement of the problem
- their documentation of each of the stages in the four-step model
- all of their trials or attempts.

The teacher should enter written comments on the student's work in an appropriate section of each problem page.

By recording their attempts, students will:

- improve their communication skills in mathematics
- provide evidence of their work and thoughts as they proceed through a problem solving process.

Some advantages for students include the following:

- they are forced to progress in a systematic way through the four-step model
- they will have a record of their attempts at solving problems over a period of time
- they will have to think clearly about what they are doing, why, and where it is taking them
- they will be able to communicate their discoveries, questions or problems to others in a small group or presentation format
- they will receive credit for their efforts.

For teachers, the advantages include the following:

- teachers will have at their disposal a tool with which to evaluate students' work and progress in solving problems
- they will be able to collect students' work, as time permits, write comments and provide feedback to individuals on a regular basis.

Students should be encouraged to keep a record of new mathematical terms or expressions. By establishing a glossary listing in their books, they will have at their fingertips a useful resource in helping to understand or clarify problem conditions.

Peer Teaching

Students who are successful problem solvers can be asked to present their solutions to other students. This not only enhances the development of the presenter's communication skills but also reinforces the processes used to find a solution. For the students who are being taught, peer teaching demonstrates that a problem's solution is attainable and provides them with instruction which may be more easily understood than that given by a teacher. Furthermore, this technique helps instil confidence in students about their ability to solve problems, and creates an awareness of other ways to solve or think about problems for students who may also have attained a solution.

Enrichment or Challenge Sessions

It may be possible to provide students with extra time to spend entirely on problem solving outside of the normal class periods. Highly motivated students may challenge one problem or a variety of problems either cooperatively or in competition, singly or in groups, depending

on the nature of the group of students and the outcomes desired by the teacher. This may be an excellent time to present a project problem with the goal to present at the end of the session as many solutions as possible, discuss procedures and strategies used and tie together the various solutions with the underlying mathematical concepts.

Clubs, Leagues and Competitions

There are many regularly scheduled mathematics contests such as the Canadian Mathematics Competition and the Alberta High School Mathematics Prize Exam. These provide students with an opportunity to display their skills in competition with others throughout the province and country. In most cases it is possible to obtain copies of questions previously used to provide practise and preparation for the competition. This preparation can be done within a regularly scheduled mathematics club or within an enrichment session such as that described on page 7.

Various mathematics leagues are in existence and are constantly looking for new teams. Some of these leagues are international, some provincial or national and a few jurisdictions have local leagues. By entering a team from a school or math club, an incentive is provided for students to solve problems; students can work cooperatively in preparing for the contests and the leagues will provide the stimulation of competition against students with like abilities and interests.

Some of the clubs and competitions available in Canada are listed in Appendix A. Various Alberta jurisdictions have their own local competitions and will supply information as to date, format and intended participants prior to the contest date. Feel free to contact the Mathematics Supervisor or Contest/League Chairperson in your area for additional information.

The NCTM has published a booklet entitled Mathematics Contests: A Handbook for Mathematics Educators, by David R. Johnson and James R. Margenau. It is a sourcebook which outlines everything from how to organize and conduct a contest to sample questions, as well as providing a listing of mathematics contests by NCTM region. It will prove invaluable to anyone who is contemplating setting up a contest or a league and is available from NCTM at the address listed in Appendix B.

Bulletin Boards

Classroom bulletin boards can be used in a variety of ways to draw attention to problem solving and provide encouragement and motivation to students to solve problems. The bulletin board can be the place where the "Problem of the Week" is posted and where correct and/or unique solutions to previous problems can be displayed. A student's self-concept is improved by having work displayed, even if it is being recognized only as a notable try or a worthy effort.

Bulletin boards can also serve as excellent sources of problem solving suggestions. Colourful posters with possible problems, strategies to try, and solutions, are readily available from school supply catalogs, as are posters with complicated graphics and "classic" or unsolvable problems and paradoxes, and biographies of the persons who posed them.

THE ROLE OF THE TEACHER

Teachers must begin to address new ways of stimulating students to "think about thinking". A list of possibilities for more effectively enhancing student capabilities include:

For Teachers

Teachers should work toward

- developing confidence by working on interesting and significant mathematical problems themselves
- being enthusiastic and demonstrating a belief in the importance of acquiring good problem solving skills
- developing and using effective questioning techniques
- creating a positive climate and non-threatening atmosphere in which problem solving can take place
- seeking diverse instructional strategies, materials and resources
- participating in professional development activities through inservice, conventions, membership in professional organizations
- evaluating their own performance and sharing experiences with their peers.

For Their Students

Teachers must learn to

- foster in students a willingness to persevere in their attempts
- reward students for their efforts and attempts
- arouse an interest and curiosity in their students so they will want to become involved in problems
- recognize and reward different ways of solving problems
- promote the documentation and communication by students of their thoughts and problem solving processes
- evaluate students' progress and achievement in a number of different ways.

Within Their Classes

Teachers should

- expose students to many varied problems
- teach students a variety of problem solving strategies, plus an overall plan of how to go about problem solving
- give students many opportunities to structure and analyze situations that really constitute problems, not just exercises
- encourage students to solve different problems with the same strategy and to apply different strategies to the same problem
- have students determine the question to be answered, select specific information necessary for solution, and choose the appropriate process. Discuss why that process is appropriate: emphasize what needs to be done and why it needs to be done rather than just obtaining an answer
- have students generalize and see similarities across problems; analyzing the structural features of a problem rather than focusing only on details
- provide sufficient time for discussion, practice, and reflection on problems and problem solving strategies
- have students estimate answers and test the reasonableness of answers
- encourage students to look back to reconsider their own thinking and note how they might have solved the problem differently.

READING AND WRITING IN PROBLEM SOLVING

The language arts program for Alberta high schools has identified five strands that are designed to develop student proficiency in communication. These are writing, reading, listening, viewing, and speaking. Certainly one of the goals in teaching problem solving in mathematics is to help students learn how to write and present their solutions to problems. In order to attain this goal, teachers must devote time to developing students' mathematical communication skills.

Reading, Listening, Viewing

Mathematics has a unique language and style all its own. To become proficient in its use, students should be given ample opportunity to experience the language of mathematics through reading, listening, and viewing. In the area of problem solving, it is important for students to read, listen to, and view good solutions to problems in order to enhance their own problem solving skills and presentations. Some possible activities are:

- reading textual and reference material
- short research assignments which involve the gathering of information on some topic of interest to the student, either as an extension from within the curriculum or from outside the curriculum, such as a famous mathematician, or a famous problem in mathematics
- evaluating presentations by other students
- viewing media such as "A Mathematical Mystery Tour" or "The Mechanical Universe".

Writing, Speaking

Writing a good solution is an integral part of the problem solving process. In fact, the true test of students' understanding of a concept or a solution to a problem is whether or not they can communicate it to someone else. The act of writing or speaking requires that students formulate, organize, internalize, and evaluate their work; that is, a high level of understanding facilitates good communication. Some possible activities are:

- have students summarize a topic in their own words
- have students present their solutions to problems to the class
- post solutions on a bulletin board
- have students maintain a journal of their successful and unsuccessful attempts at solving problems
- have students rewrite problems and make up new problems.

As is mentioned in the section on evaluation, it is worthwhile for students to write about and discuss what they are doing as they solve problems. Can they communicate their thinking processes and then evaluate and define them? Can they examine how they feel while solving problems and use this analysis to promote positive responses to the process? Students who learn to write in this manner will be able to refine their skills and become more effective and efficient problem solvers.

TEACHING CORE TOPICS USING PROBLEM SOLVING

The teaching of the four-step problem solving model can be incorporated into the teaching of the core topics. Four distinct reasons for using this approach while developing the core concepts are listed.

- The four-step model can be developed without requiring additional time or the cutting of topics.
- It assures the student that the process is a bona fide one that should be used regularly.
- It assures the student that the process is an essential part of mathematics.
- It helps to ensure that the model becomes a natural part of the students' operational mode.

Examples from three aspects of the core program are provided.

- Reviewing of previous topics.
- Developing new topics.
- Solving traditional word problems/ applications.

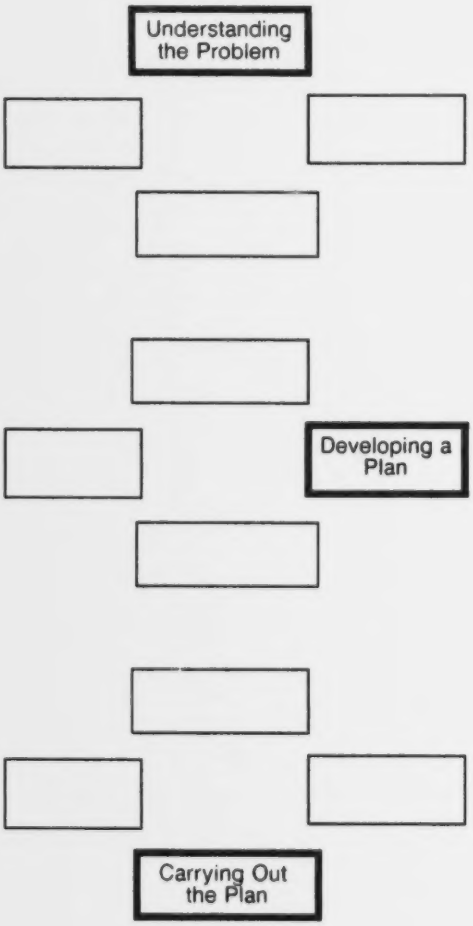
Two notes about the examples. First, the topics in which the four-step model is used have been carefully selected. Each example has a distinct purpose, such as to illustrate or practise a particular strategy, to illustrate how the model can be used for a particular purpose or simply to illustrate its usefulness in solving a problem. It is not recommended that this become the only method used in the classroom, but rather that this method be added to the variety of developmental teaching strategies already employed to meet the needs of all students.

The second note is in reference to the sample questions and answers. The examples have been provided to illustrate some of the clues and hints that teachers might give to help the class start with what they already know and proceed to build new concepts and skills. The questions are generally open-ended and are models of what the students might ask themselves. Since it is desirable that the students choose the direction to take, it is recommended that the teacher resist the temptation to provide too much structure and direction. The students need to learn how to be creative in choosing how to proceed. This will require discretion on the part of the teacher.

Reviewing of Previous Topics

The review of topics is generally a drill and practise activity wherein a student tries to recall and apply an algorithm or skill which has been learned previously. However, some objectives lend themselves to using a problem solving approach, which will reinforce the process represented by the formula. An example of this would be in reviewing how to find the surface area of a cylinder.

> Calculate the surface area of a cylinder with a radius of 14 cm and a height of 23 cm.



Understanding the Problem

Can I simulate the situation to help me understand the problem?

Will making a drawing help me to understand the problem?

Developing a Plan

Can I simulate the problem?

Can I draw a diagram?

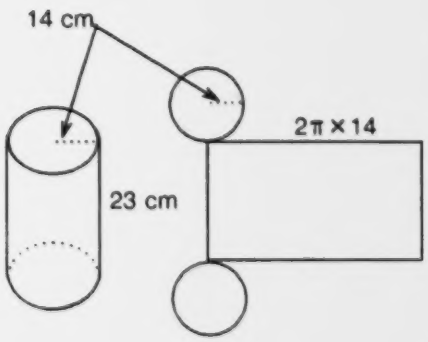
Can I partition the problem and solve each part?

What do I know that will help?

Carrying Out the Plan

Where should I start in carrying out the plan?

Use a tomato juice tin.
Make a cylinder and two circles from paper.
Draw a diagram.



How can I break the problem into simpler parts?

It is necessary to find the area of the two circles and of the rectangle.

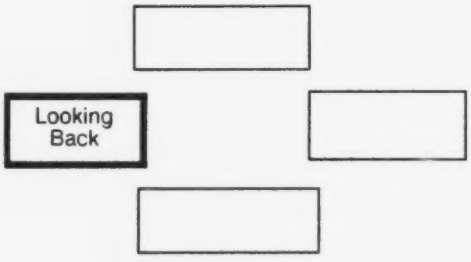
How can I find the area of the two circles and the rectangle?

The area of the two circles and the rectangle is $2\pi(14)^2 + 2\pi(14)(23)$ or 3255 cm^2.

Looking Back

What is the answer to the question?

The surface area is 3255 cm^2.



Can I generalize the solution?

In words:

The surface area of the cylinder is the sum of the areas of the two ends and the lateral surface area.

By formula:

$A = 2\pi r^2 + 2\pi rh$

Can I simplify?

$A = 2\pi r(r + h)$

Can I make and solve a similar or related problem?

Find the surface area of
- rectangular solids
- pyramids

Developing New Topics

An excellent chance to use problem solving techniques exists when new topics are introduced. Students should be encouraged to develop new concepts and algorithms by using previously acquired knowledge, skills, concepts and procedures. A good example would be the development of the formula for the sum of an arithmetic series.

Find the sum of the arithmetic series 1 + 4 + 7 + ... to 100 terms.

Understanding the Problem

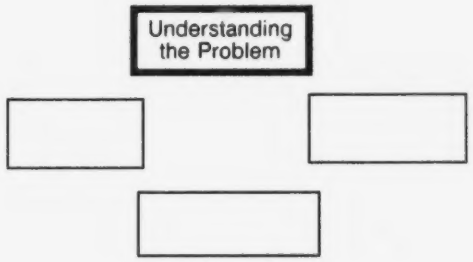
Do I know the meaning of the key words: sum, arithmetic series, and terms?

Is there sufficient information to solve the problem?

What information do I know already that would help me with this problem?

Is there a pattern to the problem?

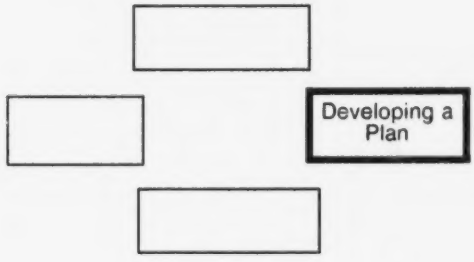
Is this problem like any I have seen before?



Developing a Plan

Can I solve a simpler problem by using smaller numbers?

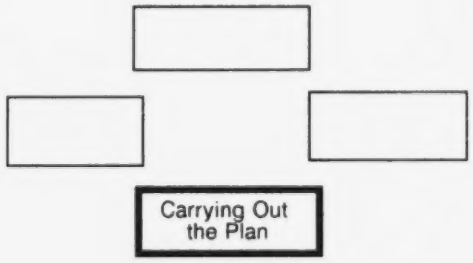
Can I apply the discovered patterns?



Carrying Out the Plan

Can I use a simpler problem?

a. 1 + 4 + 7 = ??
   What is the average of the first and last term?
b. 1 + 4 + 7 + 10 = ??
   What is the average of the first and last term? What is the average of the second and second last term? How many averages are there?
c. 1 + 4 + 7 + 10 + 13 = ??
   What is the average of the first and last term? What is the average of the second and second last term? How many averages are there?
d. Repeat part c as necessary for 6,7,... terms until students see that the sum of the first and last terms divided by 2 gives the average and that there are *n* of these averages.



What about applying the pattern?

The sum of five terms is found by taking five times the average of the first and last terms.

Can I apply this to the original question?

The sum of 100 terms is found by multiplying the average of the first and last terms by 100.

Do I have enough information to find this product?

I do not know the last term.

How can I find that term?

I can apply the formula I used when I studied arithmetic sequences yesterday, $a_n = a_1 + (n-1)d$.

The 100th term is 1 + (99)3 or 298.

Can I find the sum of 100 terms now?

Yes, it is 100 times the average of the first term (1) and the last term (298), which is 14 950.

Looking Back

What is the answer to the question?

The sum of 100 terms of the series is 14 950.

Can I generalize the answer?

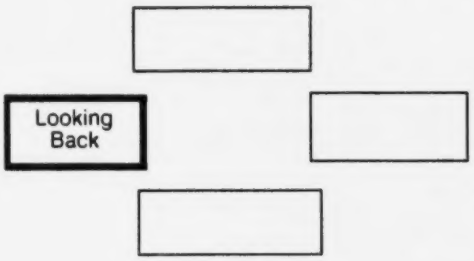
$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

What other methods are there of doing the problem?

a. Find the sum of the series and the reverse of the series; i.e.,

$$\begin{array}{l} 1 + \quad 4 + \quad 7 + \ldots + 295 + 298 \\ \underline{298 + 295 + 293 + \ldots + \quad 4 + \quad 1} \\ 299 + 299 + \ldots \qquad\qquad + 299 \end{array}$$

b. Use symbols for the terms; i.e., a_1, $a_1 + d$, ..., $a_1 + (n-1)d$ and find the sum in the same manner as part (a) above.



Solving Traditional Word Problems/ Applications

Traditional word problems provide opportunities to apply problem solving processes. Students can use previously acquired algebraic techniques in conjunction with problem solving strategies to find solutions.

> Sara is 4 a older than her brother Hanif. The sum of their ages is 7/10 that of their mother's age. In 12 a time the sum of their ages will equal their mother's age. Find Sara's, Hanif's, and their mother's present age.

<u>Understanding the Problem</u>

Have I read and reread the problem carefully?

What am I being asked?

Is there a pattern to the problem which might help me to understand?

Do I understand the relationship in the problem (i.e., who is older; how the mother's age compares to the childrens; and the comparison of ages in 12 a)?

<u>Developing a Plan</u>

Can I see a relationship in the problem by using a chart?

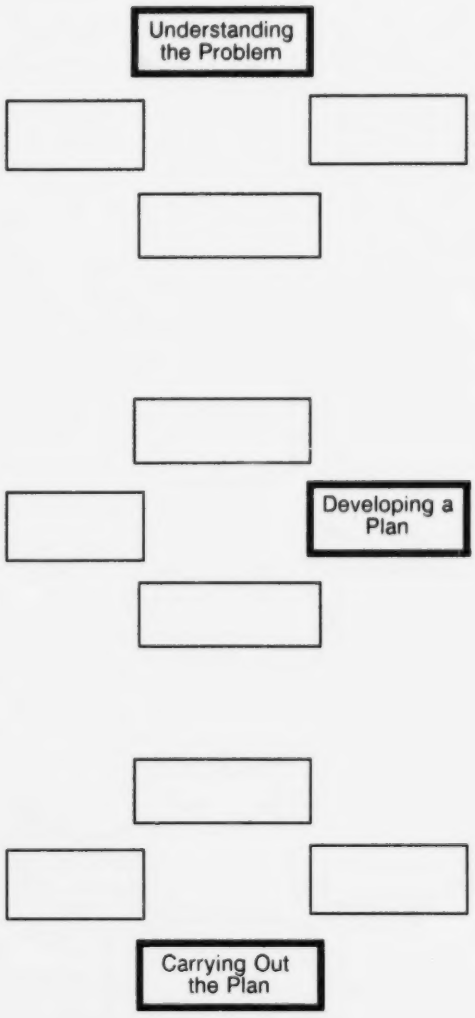
Can I formulate an equation to show this relationship?

Can I make an intelligent guess, check, refine, and recheck?

<u>Carrying Out the Plan</u>

Where do I start in carrying out the plan?

A relationship between the ages of the children and their mother is given both at the present time and in 12 a. Summarize the information in chart form.



	Hanif	Sara	Mother
Present ages	x	$x+4$	$\frac{10}{7}(x + x + 4)$
Ages in 12 years	$x+12$	$x+16$	$\frac{10}{7}(2x + 4) + 12$

How can I consolidate the information in the chart?

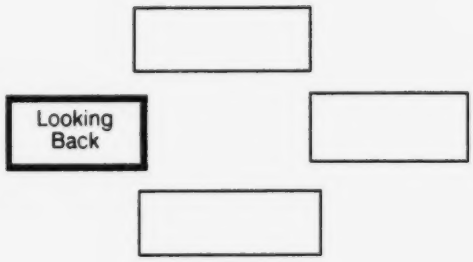
Write an equation using the ages in 12 a and solve the equation for x which will give us Hanif's present age.

$$(x+12)+(x+16)=\frac{10}{7}(2x+4)+12$$

$$2x+28=\frac{20}{7}x+\frac{40}{7}+12$$

$$14x+196=20x+40+84$$
$$72=6x$$
$$x=12$$

Looking Back



Have I answered the question asked in the problem?

Since x represents Hanif's present age, Hanif is 12 a old, Sara is 16 a and mother is 40 a.

Is the answer reasonable?

The children could be 12 a and 16 a if the mother is 40 a old.

Have I checked the answer?

At present, the sum of Hanif's and Sara's ages is 28 which is 7/10 of their mother's age (40). In 12 years Hanif will be 24, Sara will be 28 and mother will be 52, which will be the sum of Sara's and Hanif's ages.

Can I solve the problem in another way?

I could make a reasonable guess, check and refine the answer.

Can I make and solve a similar or related problem?

For any age problem I could make a chart showing the relationship of the ages, and then write an equation.

Can I solve similar problems such as distance, money, or mixture problems?

By setting up a chart to summarize the problem and then writing an equation to show the relationships in the chart, other problems of this type can be solved.

The key to using a problem solving approach in the presentation of core topics is to ask good questions which will help students choose a direction to achieve a teacher defined goal. These questions should serve as models for ones which students will ask themselves as their facility for problem solving grows.

EXTENDING AND CREATING PROBLEMS

The solving of problems has been built on Polya's four-step model which involves

1. Understanding the problem
2. Developing a plan
3. Carrying out the plan
4. Looking back

Many teachers feel that the steps of developing a plan and carrying out the plan are the most important ones, and spend much of their time in this area. The neglected part of problem solving is the last step of looking back. In addition to looking back to see if a solution makes sense within the context of the original problem, it is also important to look ahead to extend and create more problems from the one just completed.

In many cases where students are working on problem solving you will hear the question, "What happens if...?" This is the key to extending problem solving: to let the students do the extension based on the questions they ask.

Consider the locker problem in which doors are opened and shut on lockers with the first student opening all the doors, the next student closing every second door, the third student taking every third door and changing its state; that is, opening it if it is closed and closing it if it is open. One student asked, "What happens if instead of opening every door for the first, every second door for the second, and every third door for the third, etc., the first does every door, the second does every third door, and the third does every fifth door?" This question can become the problem for the whole class or for that particular student. Don't worry that you, the teacher, may not know the answer or that you may not immediately be able to arrive at the answer. Make the solution the responsibility of the whole class or of that student. Also, students should witness an honest search for a solution that is not immediately known.

Another example of a student-generated extension occurred in a class working with pentominoes, finding different arrangements using four pentominoes to build a four by five rectangle. The question asked was, "Can we build a rectangle with any area that is a multiple of five, up to an area of 60 square units?" Even if the teacher knows the answer is yes, the response to the question should be "Let's find out."

Make use of the ideas that students have. It may be the incentive students need to work on problems of their own. It may provide incentive to other students in the class to experiment with various types of problems and to get involved in the making of mathematics.

For further information see Sawder, Larry. "The Looking-Back Step in Problem Solving." Mathematics Teacher 79:7 (Oct. 86): 511-513.

EVALUATING PROBLEM SOLVING

WHY EVALUATE?

The increased emphasis on problem solving carries with it the challenge of evaluating the effectiveness of instruction. As teachers, we must be able to tell if we are being successful in achieving one of the major goals of the mathematics program. We must be able to report to the stakeholders in students' education (the students themselves, their parents, the school and district administration and the provincial government), how effectively and efficiently students are able to problem solve.

Problem solving is a process. For this reason it is necessary to evaluate how successful students are at all stages of the process. Just getting an answer is not enough; in fact problem solvers know that that is only one part of solving a problem. For this reason, evaluation of problem solving must deal with all aspects of the process and the success students are having in carrying out all stages of the process.

WHAT TO EVALUATE

Evaluating the problem solving process should involve the important goals of teaching problem solving. Problem solving is a very complex activity. It involves the use of a variety of skills and strategies, and the ability to evaluate one's own thinking and progress. Success in problem solving is the result of the problem solver's interest, motivation and confidence. The teaching of problem solving should enable students to acquire the following:

1. Problem solving skills such as
    - understanding the question asked
    - understanding the conditions of the problem
    - selecting data needed to solve the problem
    - selecting the appropriate solution strategy or strategies to pursue
    - implementing the solution strategy
    - answering the question asked
    - evaluating the reasonableness of an answer.
2. Abilities developed to select and use problem strategies such as
    - mastering a repertoire of strategies for solving problems
    - knowing the conditions for when and how to use them.
3. Abilities to use related knowledge such as
    - recalling and applying specific knowledge
    - determining when to use it
    - applying the newly acquired mathematics skills.
4. Abilities to monitor and evaluate thinking and progress by answering questions such as
    - What am I trying to do?
    - What have I done?
    - What still needs to be done?
5. Abilities to get the correct answers to a variety of problems and to make a statement that answers the question asked.
6. Abilities to solve problems cooperatively such as
    - developing social skills
    - clarifying one's ideas by sharing them with others
    - evaluating another person's solution
    - comparing and discussing solutions.
7. Attitudes and beliefs about problem solving such as
    - that problems can be solved in more than one way
    - that problems may have more than one answer
    - that perseverance pays off
    - that trying another strategy may be helpful.

HOW TO EVALUATE

The goals for teaching problem solving suggest that it is important to evaluate student progress in problem solving skills and strategies, and attitudes and beliefs regarding problem solving. The purpose of this section is to describe several techniques for evaluating these outcomes.

1. Observation and Questioning Techniques

The observation and questioning of students while they are in the process of solving problems can provide invaluable insights into their thinking processes, attitudes and beliefs. Observation can be done by two methods.

a. Informal Observation and Questioning

An environment should be created where students are not distracted or inhibited by the teacher's presence. The observation should be focused so that the teacher looks specifically at those aspects of problem solving which cannot be evaluated by other means. However, it should be flexible enough to allow other significant observations to be noted. Observation should occur with only a few students at a time.

In questioning students about their problem solving skills and attitudes, it is necessary to ask questions that focus on the process being used as well as their feelings about problem solving. Questions must be open-ended to allow the students to answer without being directed to an answer.

A technique should be developed to record the results of the observation and questioning as quickly as possible. It could take the form of a checklist, a comment card or a rating scale.

Informal observation may be the best technique of evaluating the thinking processes while solving problems. It can assess the attitudes and beliefs of the students, such as perseverance in solving problems and willingness to try problems, something which cannot be done easily by other techniques.

b. Structured Interviews

In a structured interview individual students are asked a pre-selected set of questions while solving a problem. The questions should probe the student's problem solving skills to see how well they understand the problem, how they select and use data, how they select and use a strategy, how they answer the problem, and how they check on the reasonableness of their solution. As in the informal interview, some form of recording technique should be used to summarize the results of the interview.

2. Holistic Scoring Techniques

Evaluation of a student's written work has typically meant that the teacher will look first at the answer, and if it is incorrect, check the process to find the cause of the error. With the emphasis on problem solving, evaluation of the process becomes the primary focus. Three methods for evaluating solution processes are outlined here.

a. Analytic Scoring

This is an evaluation technique in which a range of scores is assigned to work completed. A score of zero would indicate a complete absence of the behaviour or procedure being analyzed. A score at the top of the range would indicate that the student has complete understanding of the process or has exhibited exemplary skill in the behaviour under scrutiny. The type of process being evaluated and what the teacher perceives as the important phases of that process will

be the basis upon which a range of scores is decided. For example, if a teacher wished to evaluate the "Understanding the Problem" phase of a student's solution, the range might be defined as follows:

0. Completely misunderstands the problem.
1. Misinterprets part of the problem but demonstrates some understanding.
2. Demonstrates understanding of the problem.

In the "Developing a Plan" phase, the teacher may decide on a four-point range (0-3) with the intermediate steps reflecting levels of elegance or ingenuity.

An analytic scale provides a means of assigning a numerical value to a student's written work. It does, however, have the disadvantage of being unable to provide information about the thinking processes which went into the solution. As well, students with equal total scores may have very different strengths and weaknesses (2-1-1-1 is very different from 3-2-0-0). Such differences must be kept in mind by teachers; they show that analytic scales should be used in conjunction with other techniques.

b. Focused Holistic Scoring

This method assigns a single score to the entire solution of a problem. Specific criteria are established and the work is evaluated as to how it matches the criteria. Again the range is established by the teacher based on the type, difficulty and complexity of the problem. Scores would range from 0 (not scorable) to an established maximum which would indicate excellence and correctness.

This method of evaluation provides a fairly quick assessment while providing a numerical value based on pre-established criteria. However, it does not provide much information about a student's thinking processes nor does it identify a student's particular weaknesses.

c. General Impression Scoring

In this technique the teacher gives a rating based on a general impression of the solution of the problem. As it depends on implicit criteria on the part of the evaluator, this procedure should be used infrequently and only by those who have much experience in assessing problem solving. It should not be the only method used and should be accompanied by anecdotal comments and questions.

Although this method is quick and easy, its main downfall is that it is hard to defend a mark if no explicit criteria are listed. For this reason, its usefulness in reporting to students or parents is severely limited.

3. Using Multiple Choice and Completion Tests

The evaluation of students in problem solving is most often done by a written test. A written test helps answer two important questions:

- Can the student carry out the various thinking processes involved in solving a problem?
- Can the student find the correct answer?

Since most tests provide the answer to the second question, a written test should also be able to provide answers to the first question.

In preparing a test for students, clear-cut objectives of the problem solving process must be established and appropriate questions developed to test each of these objectives.

For example, under "What to Evaluate", item 1 "evaluating the reasonableness of

an answer": a specific multiple choice question to test this goal could be:

> Which statement provides the most reasonable solution to the problem?
>
> PROBLEM: The radius of the earth at the equator is 6400 km. What is the circumference of the earth at the 45th parallel of latitude?
>
> The circumference of the earth at the 45th parallel of latitude is
>
> a. 20 000 km
> b. 4500 km
> c. 28 000 km
> d. 40 000 km

The use of tests to evaluate problem solving is suitable for evaluating performance but not for evaluating attitudes. Therefore, other types of evaluation should be used in conjunction with written tests to evaluate problem solving.

4. Using Self-assessment Data from Students

Evaluating the achievement of some of the goals of problem solving is best done through self-assessment by students. By evaluating and reporting on their own activities, they will be able to see more clearly the processes through which they work while solving problems.

a. Student Self-reports

Students can write or dictate a report based on questions provided by the teacher regarding specific aspects of a problem solving activity. Such reports should give a description of the processes used to attain a solution and should help students concentrate on the thinking they did in pursuit of a solution. Students can also identify particular strategies and previous experiences applied in solving the problem. This technique can be used as a regular part of the "Looking Back" phase of the problem solving framework.

This procedure provides information about a student's thinking which is unavailable through other means and allows a student to feel a part of the evaluation process. It does, however, require a great deal of student time and should not be used for the purpose of assigning marks.

b. Inventories

An inventory is a checklist which students may use to evaluate attitudes toward problems and processes used in their solutions. An inventory may be one such as that developed for the Mathematics Problem Solving Project at Indiana University (see Charles et al, 1987 "How to Evaluate Problem Solving") which had twenty points testing the categories of willingness to engage in problem solving activities, perseverance during the process and self-confidence with respect to problem solving.

This technique also has the advantage of gathering student input but it does not take as much time as a report. However, it should be used with other methods as its validity and reliability may not be consistent.

Conclusion

In establishing a procedure for the evaluation of problem solving, Charles, Lester and O'Daffer (1987) suggest that certain guidelines should be taken into consideration:

1. Classroom evaluation should be regular and systematic.
2. Thinking processes as well as correct answers should be evaluated.
3. Evaluation plans should match the instructional goals.

4. Attitudes and beliefs about problem solving should be assessed as well as performance.

5. Observations of students' small group efforts and written work should be used as an important part of the evaluation plan.

6. Students should be interviewed individually whenever possible.

7. It is not necessary to evaluate all students at the same time, nor is it necessary to evaluate their performance on every problem solving experience.

8. Students should be informed of your evaluation plan and how it works.

We acknowledge with thanks the work of Randall Charles, Frank Lester and Phares O'Daffer in How to Evaluate Progress in Problem Solving, National Council of Teachers of Mathematics (1987), and Charles Hucka of the NCTM for granting permission to use the material in this form.

PROJECT PROBLEMS

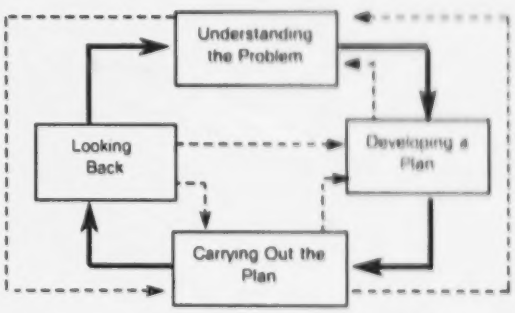
Some problems are rich. They are rich because they can be solved in a variety of ways, and in the process of solving them they suggest additional questions to be investigated. We call such problems "project problems" because the original problem leads to new problems, and in the process a number of different mathematical topics may become involved. One example of a project problem is given at the right.

> Can the number 525 be written as the sum of consecutive numbers? In how many different ways can this be done?

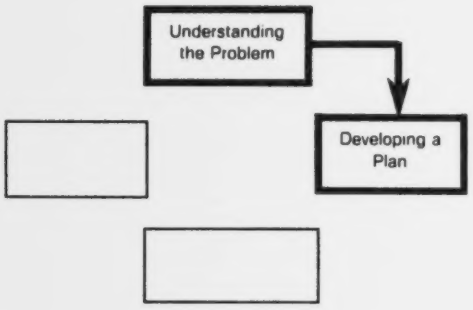
This problem is suitable for a wide range of students. It can be solved by students in the upper elementary grades, using trial and error strategies and thinking in terms of division, averages, and even and odd numbers. With students in junior high school these ideas may lead to a consideration of the possibility of negative numbers, and an interesting extension of both the method and the number of solutions. For students in the senior high grades, more sophisticated strategies may be expected, and excursions into topics such as arithmetic series, quadratic equations, prime factorization, and enumerating the subsets of a given set are possible. Solutions which involve the writing of computer programs are also worthwhile and accessible, even for students with limited programming abilities.

Before reading the discussion that follows, the reader should spend some time working on the problem and considering the problem solving strategies and mathematical concepts that students at various levels might call upon as they tackle it.

In the discussion of this problem we will make reference to the model of the problem solving process shown at the right. We will indicate how this model's stages and transitions between stages are exemplified by the work that problem solvers engage in as they attempt to solve the problem, as they investigate alternative ways of approaching the problem, and as they set and solve related problems suggested by their reflections.



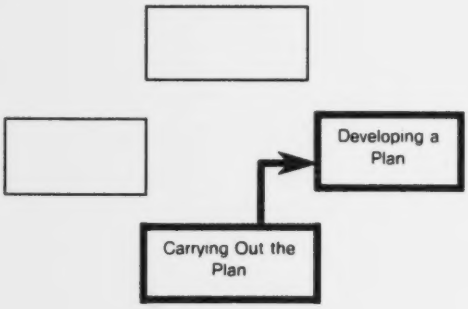
Perhaps the least sophisticated approach to this problem is a brute force frontal attack. Many people think of consecutive numbers as being like counting, and think immediately of starting from 1. But to add 1 + 2 + 3 + ... trying to reach 525 is not exactly easy. Even with a hand calculator there is a good chance of error, since there will be more than 30 terms in the sum. A more advanced trial and error procedure would be to write a computer program, such as the one on the right in BASIC.



Understand: Consecutive numbers are 1, 2, 3, ... and so on. Plan: Add 1 + 2 + 3 +...

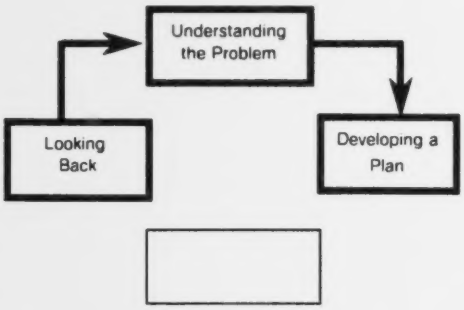
```
10 LET S = 0 : LET N = 1
20 LET S = S + N
30 PRINT N, S
40 IF S > = 525 THEN END
50 LET N = N + 1
60 GO TO 20
```

If we run this program or do the computation on a calculator, we find that the sum of the first 31 consecutive numbers is 496, but the sum of the first 32 is 528. So it may be tempting to conclude that the answer to the question is no, 525 cannot be written as the sum of consecutive numbers. Yet the wording of the question suggests that the answer is yes and that more than one solution is possible. This line of thinking suggests that the solution involves starting somewhere other than 1, and perhaps finding the sum of 525 with different numbers of addends (terms).



New Plan: Write a computer program

One variation of the trial and error approach is to use successive approximations. In this case, since 1 + 2 + ... + 31 + 32 = 528, and 525 is close to 528, perhaps we can "adjust" our first attempt in order to arrive at the proper sum. Since the sum is too big by 3, we can subtract 3 by omitting the first two terms. Thus we see that 3 + 4 + ... + 31 + 32 = 525 will be a solution. Changing line 10 in the program above to "10 LET N = 3" demonstrates this nicely. Although this gives us one solution, it does not suggest how to find other solutions, except by trying other starting numbers. The next largest starting number that works is 9 (i.e., 9 + 10 + ... + 32 + 33 = 525).

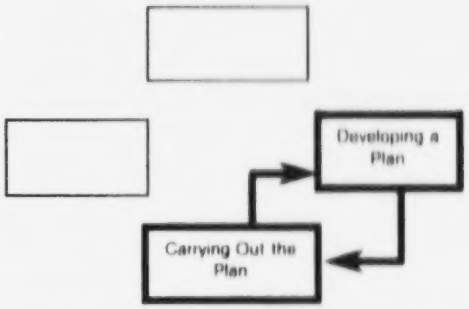


Look Back: No answer seems possible
Understand: Problem suggests more than one answer is possible
New Plan: Adjust first attempt

A second modification of the trial and error approach might involve systematically varying the number of addends in the sum. This amounts to asking "Can we find two consecutive numbers that add up to 525? Three consecutive numbers? ... etc." Since the two numbers we are looking for are almost the same size, we can divide 525 by 2 to see about how big they are. The quotient is 262.5, so we know the two numbers will be 262 and 263; 262.5 is their average. Similarly, 525 divided by 3 is 175, so the three numbers are 174, 175, and 176. When we try 525 divided by 4 we get 131.25, which is neither a whole number nor halfway between two whole numbers, so there are no four consecutive whole numbers that add up to 525. This sort of systematic trial and error approach can be summarized in a table such as the one to the right.

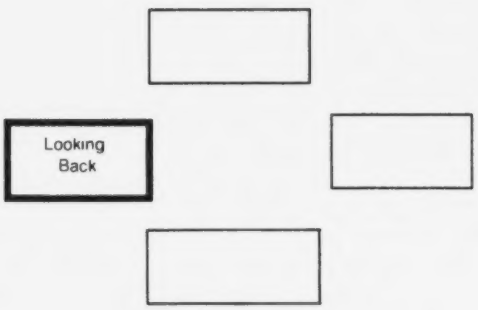
Of course this table can be continued, but several features of the table so far are worth discussing. Following up on the idea of averages, we note that for each term that is less than the average there is a corresponding term that is greater than the average. We might ask "Do there appear to be any patterns in the table?" In each case where there is a solution with an odd number of terms, the middle term is the average. Where there is no solution with a proposed odd number of terms, 525 divided by the given number of terms is not a whole number. Where there is a solution with an even number of terms, 525 divided by the number of terms is "something and a half" (e.g., 525 ÷ 6 = 87.5), and the two middle terms of the sum are found on either side of this number (in this case 87 and 88).

We might conjecture from this table that there are no solutions with a number of terms that are divisible by 4, since neither 4 terms nor 8 terms is possible, and neither is 12 or 16 terms (because 525/12 = 43.75 and 525/16 = 32.8125). This conjecture is in fact true, and can be proved by noting that if the number of terms is divisible by 4 there will be an even number of pairs of consecutive numbers, such as (a + a+1) + (a+2 + a+3) in the case of four terms, (a + a+1) + (a+2 + a+3) + (a+4 + a+5) + (a+6 + a+7) in the case of 8 terms, and so on. Now whether *a* is even or



New Plan: Find solutions with 2 addends, 3 addends, ... etc.
Continue the Plan: Systematically vary the number of addends

NUMBER OF TERMS (N)	AVERAGE (525/N)	MIDDLE TERM(S)	SOLUTION 525 = ...
2	262.5	262, 263	525 = 262 + 263
3	175	175	525 = 174 + 175 + 176
4	131.25	?	Impossible
5	105	105	525 = 103 + 104 + ... + 107
6	87.5	87, 88	525 = 85 + 86 + ... + 90
7	75	75	525 = 72 + 73 + ... + 78
8	65.625	?	Impossible
9	58.33	?	Impossible
10	52.5	52, 53	525 = 48 + 49 + ... + 57



Look Back: What patterns are there in the table of solutions so far?

odd, each of the sums of two consecutive numbers will be odd, and the whole sum will be the sum of an even number of odd numbers, which will be even. Thus the entire sum which is even cannot be 525 which is odd. While this argument sounds convoluted, the hint that we should consider whether the sum of 4 consecutive numbers, 8 consecutive numbers, and so on is even or odd, together with some questions about why this is so, can lead students to this discovery and to an understanding of why it must be true.

When the table is continued, we find that the next few solutions are those shown in the table to the right. In order to avoid having to check all values of N (the number of terms) we might ask "What are the properties of N in the cases where there is a solution that has N terms?" We noted before that if the number of terms (N) is odd, it must divide 525 evenly, and that if N is even, 525 /N must be "something and a half". This is equivalent to saying that N must be a factor of 1050 (twice 525). In fact, this is the key to solving the problem (whether we realize it at this point or not); there are as many ways of writing 525 as the sum of consecutive numbers as there are factors of 1050.

An awareness of symmetry is often useful in solving problems, for recognizing when we have come halfway often means that the remaining half is in some sense "just like the first half". We might conjecture that since we have a solution with 21 terms centred on 25 and another with 25 terms centred on 21 we have reached the "halfway point" and that the remaining solutions correspond somehow to the previous ones. This conjecture, unfortunately, is not quite correct. To see why, consider what solution would correspond to the solution with 14 terms whose average is 37.5. Could it be 37.5 terms that have an average of 14? Clearly not. The symmetry theme does not yet seem helpful, but we will consider it again later.

If we continue the table, we find that the next factor of 1050 after 30 is 35. Following the pattern we have established, we find that for 35 terms the average addend must be 525/35 = 15, so we look for 17 consecutive numbers up to 15, then 15, then 17 consecutive numbers after 15, for a total of 35

Look Back: Are there any solutions in which the number of terms is divisible by 4? Why, or why not?

Look Back: What are the properties of the number of terms?

NUMBER OF TERMS (N)	AVERAGE (525/N)	MIDDLE TERM(S)	SOLUTION 525 = ...
14	37.5	37,38	31 + ... + 37 + 38 + ...+44
15	35	35	28 + ... + 35 + ... + 42
21	25	25	15 + ... + 25 + ... + 35
25	21	21	9 + ... + 21 + ... + 33
30	17.5	17,18	3 +...+ 17 + 18 + ... + 32

32

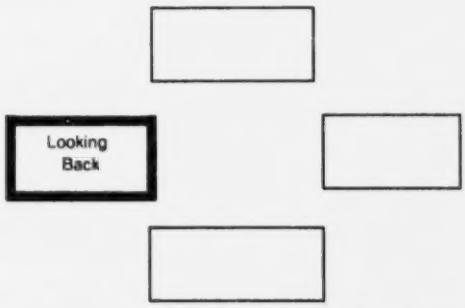
terms. But how can this be? How can there be 17 different consecutive numbers that are less than 15? A knowledge of negative integers helps us overcome this apparent contradiction. We see that the solution with 35 terms is the following:

$(-2)+(-1)+0+1+\ldots+14++15+16+$
$17+\ldots+31+32=525.$

Returning to the theme of symmetry and correspondence, we ask "Does this solution correspond to any previous solution?" and we see that it is the same as the 30-term solution 525 = 3 + ... + 32 except that it has five additional terms whose sum is zero, namely (-2) + (-1) + 0 + (+1) + (+2). Using this correspondence, we can find another solution (beginning with negative integers) by adding [-(a - 1) + -(a - 2) + ... + 0 + ... + (a - 2) + (a - 1)] to each previous solution, where *a* is the first term of the previously found solution. Thus for each solution above the heavy line in the table to the right, there is a corresponding solution which begins with negative integer terms.

Returning to the symmetry theme, we now see that the 35-term solution corresponds to the 30-term solution, the 42-term solution to the 25-term solution, and so on. So the 14-term solution (with average term 37.5) corresponds not to a 37.5-term solution but to a 75-term solution whose addend average is 7 (not 14).

Our former conjecture involving corresponding the numbers of terms was almost correct. Instead of looking for a product of 525, we should have been looking for a product of 1050, because the number of terms can be even as well as odd.



Look Back: Are there patterns in the solutions found so far that can be used in the remaining solutions?

Look Back: Do some solutions include negative integers? How are these solutions related to solutions without negatives?

NUMBER OF TERMS (N)	AVERAGE (525/N)	MIDDLE TERMS(S)	SOLUTION 525 = ...
2	262.5	262, 263	262 + 263
3	175	175	174 + 175 + 176
5	105	105	103 + 104 + ... + 107
6	87.5	87, 88	85 + 86 + ... + 90
7	75	75	72 + 73 + ... + 78
10	52.5	52, 53	48 + ... + 52 + 53 + ... + 57
14	37.5	37, 38	31 + ... + 37 + 38 + ... + 44
15	35	35	28 + ... + 35 + ... + 42
21	25	25	15 + ... + 25 + ... + 35
25	21	21	9 + ... + 21 + ... + 33
30	17.5	17, 18	3 + ... + 17 + 18 + ... + 32
35	15	15	(-2) + ... + (+2) + 3 ... + 32
42	12.5	12, 13	(-8) + ... (+8) + 9 + ... + 33

Having come this far, let us finish, and answer the question "In how many ways can 525 be written as the sum of consecutive numbers?" We have 11 table entries above the heavy line and 11 below, for a total of 22. Is that all the solutions? Earlier we indicated that the total number is the same as the number of factors of 1050. But we do not have any solutions corresponding to 1 and 1050 which are also factors of 1050. Should we? Is there a solution with one term? That depends upon whether we consider 525 = 525 to be a way of writing 525 as the sum of one consecutive number. Is there a solution with 1050 terms? Yes, it is the following, which has 524 negative terms, zero, and 525 positive terms.

$525 = (-524) + ... + 0 + ... + 524 + 525.$

With a class studying combinatorics topics such as selections, permutations, combinations, and the enumeration of the subsets of a given set, the problem of finding all the factors of 1050 might be tackled by listing the set of prime factors of 1050 and determining how many subsets that set has.

The idea that an N-element set has 2^N different subsets is helpful, but the prime factorization $1050 = 2 \times 3 \times 5 \times 5 \times 7$ contains a repeated factor (5 appears twice). Therefore, the number of factors N(f) is

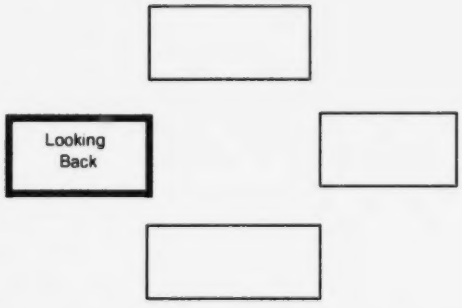
$$N(f) = 2^5 - 2^3 = 24.$$

There are 2^5 subsets altogether, but 2^3 of them, the ones having one or the other but not both of the two 5's, are counted twice, once with each of the two indistinguishable 5's.

Also there are 2^3 subsets containing no 5's, the same number containing one 5, and the same number containing both 5's, so

$$N(f) = 2^3 + 2^3 + 2^3 = 3 \times 2^3 = 24$$

is another way to find the number of factors of 1050.



Look Back: Is there a way to find all the factors of 1050? Can we tell how many there are without creating a list?

A class studying arithmetic sequences and their sums might think of this problem as an application in which a series has a sum of 525 and a difference between terms of 1. If we let the number of terms be n, and the first term be a, the difference between terms be d, and the sum be S, we have

$$S = a + (a + d) + \ldots + (a+(n-1)d)$$

which becomes

$$S = \frac{n}{2}\left[a + (a + (n-1)d)\right]$$

Substituting 525 for S, 1 for d, and simplifying, we get

$$1050 = 2na + n^2 - n \text{ or}$$

$$n^2 + (2a - 1)n - 1050 = 0.$$

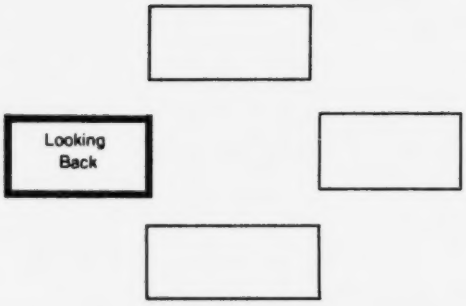
This is a quadratic equation in n (the number of terms), but it also has another variable, a (the first term of the series). The solutions we are looking for have integral positive values for n and integral values for a. From here we can proceed in either of the two ways we used before: (1) we can try various values for the first term of the series a and solve for n, or (2) we can propose values for n and use them to solve for a.

Taking the first of these methods we ask "Is there a solution whose first term is 1?" Substituting a = 1 in the equation above we get

$$n^2 + n - 1050 = 0.$$

It is easy to see that for this quadratic equation there are no integral solutions. The first value of a which does produce a factorable quadratic and therefore an integral solution is a = 3. This yields the quadratic $n^2 + 5n - 1050 = 0$ which can be factored as follows:

$$(n + 35)(n - 30) = 0.$$



Look Back: How can the mathematics of arithmetic series be applied in this problem?

These equations show that the series beginning with 3 will have 30 terms; the solution n = -35 is meaningless because we cannot have a negative number of terms. This process could be continued by trying various other values for a and finding corresponding positive value for n. The next value of a that works is 9; it produces n = 25 terms. This abstract, symbolic work is the exact parallel to the concrete trial and error approach we described earlier, wherein we used a calculator or computer program to test whether the sum of consecutive numbers starting with a given first term reached exactly 525.

In the second method we used before, we systematically tried various values for n (the number of terms) and we found the middle term or terms by dividing 525 by n. A similar approach lets us find a (the first term) by writing the quadratic

$$n^2 + (2a - 1)\,n - 1050 = 0$$

in its factored form

$$(n + p)\,(n - q) = 0$$

and noting that for any given values of p and q we can find the first term by solving for a in

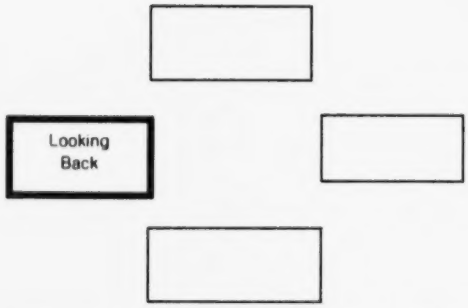
$(2a - 1) = p - q$, that is

$a = 1/2\,(p - q + 1)$.

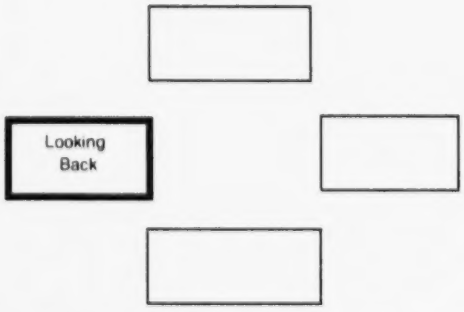
Now since p x q = 1050, p must be even and q odd or vice versa. In either case, p - q + 1 will be even, so a will be a whole number. This proves that there are as many solutions as there are factors of 1050, a fact we hypothesized earlier and proved by brute force.

One final suggestion for extending this problem centres on the idea that other problems could be made by changing the number 525 to some other value. Some questions worth exploring include the following:

- Why do you think that 525 was chosen?
- What makes it a good number?
- What are some other good numbers?
- Are there bad numbers?
- Are there any numbers which cannot be written as the sum of consecutive whole numbers?



Look Back: How do these abstract, algebraic methods compare to the concrete computational methods described earlier?



Look Back: Why was 525 used in this problem? What other numbers might be used in similar problems?

In solving this problem and reflecting on it we have brought to bear a number of mathematical concepts and skills that were not obviously involved in the statement of the original problem, and we have solved a number of related problems and sub-goal problems. In order for students to share in and benefit from a process like this, it is necessary for them to ask themselves the kinds of questions we have asked, perhaps as a result of a question or hint from the teacher. This suggests that the teacher needs to be aware of the progress that students are making on their own so that the teacher can insert a question or hint in a timely fashion. Of course, there may be other ways of solving the problem, perhaps involving other mathematical topics and techniques. This possibility also demands a high level of awareness on the part of the teacher who is attempting to promote growth in problem solving.

Certainly this is a rich problem. Whether we realize it immediately or not, many other problems prove to have this kind of richness when we take the time to follow them where they lead us. One mark of a good problem solver and teacher of problem solving is the ability to recognize and capitalize upon this richness.

OTHER PROJECT PROBLEMS

Another problem which might be looked upon as a project problem is given at the right.

> What is the probability that in a class of 30 students there will be at least two students who have the same birthday?

One way of solving this problem is to gather data. If we make copies of a sheet with the year's calendar on it and circulate them around different classes asking 30 students to circle their birthdays, we can check what percentage of these sheets have at least one date circled more than once. This solution is based on the concept of probability as relative frequency.

A different approach to solving this problem is to perform a simulation. If we write the numbers from 1 to 365 (or 366, including February 29) on cards or slips of paper and put them into a container, we can perform an experiment by mixing the cards, drawing one out, writing down its number, returning it to the container and repeating this process 30 times. We check the list of numbers for repeats and perform the entire process a number of times.

A question that naturally arises from these methods is "how many trials do we need to be confident that the probability we have found is stable?" This is an instance of the "law of large numbers".

It is also possible to solve this problem using a computer simulation. The main components required are

- a routine that makes a list of 30 random numbers, each number in the range from 1 to 365,
- a routine that searches the list and notes whether any entries in the list are the same number, and
- a routine that repeats both of the steps above a given number of times and reports what fraction of the trials had at least one match.

The program at the right, written in BASIC does the experiment 100 times and gives the probability as a percentage.

```
10    REM  INITIALIZE
20    LET  F = 0: DIM X(30)
30    REM  BEGIN 100 TRIALS
40    FOR  K = 1 TO 100
50    REM  LIST 30 BIRTHDAYS
60    FOR  I = 1 TO 30
70    LET  X(I) = INT (RND (3) •365)
80    NEXT I
90    REM  SEARCH FOR MATCHES
100   LET  M = 0
110   FOR I = 1 TO 29
120   FOR J = I + 1 TO 30
130   IF X(I) = X(J) THEN LET M = 1
140   NEXT J
150   NEXT I
160   REM  CHECK FREQ OF MATCHES
170   IF M = 1 THEN LET F = F + 1
180   REM  CONTINUE TRIALS
190   NEXT K
200   REM  REPORT FINAL RESULT
210   PRINT "IN ";F;"% OF THE CLASSES"
220   PRINT  "AT  LEAST  TWO  STUDENTS
      HAD THE"
230   PRINT "SAME BIRTHDAY"
240   END
```

The "theoretical" or "expected" probability can also be calculated based on the assumption that each birthday is equally likely. The technique which is required involves calculating the probability that 30 people chosen at random all have different birthdays, and then subtracting this probability from one. In algebraic form that is

$$P = 1 - \frac{365 \times 364 \times \ldots \times 336}{(365)^{30}}$$

A computer program (in BASIC) that performs this calculation is given at the right.

```
10   REM PROGRAM TO COMPUTE PROB
11   REM THAT IN A CLASS OF 30
12   REM AT LEAST TWO STUDENTS
13   REM HAVE THE SAME BIRTHDAY
20   REM
21   N = NUMBER OF STUDENTS
22   REM (1, ..., 30)
23   REM
24   REM PN = PROB THAT ALL N
25   REM BIRTHDAYS ARE DIFFERENT
26   REM
27   REM P = PROB OF AT LEAST
28   REM ONE MATCH
29   REM
70   PRINT "N", "PN", "P"
80   LET PN = 1
90   FOR  N = 1 TO 30
100  LET PN = PN * (365 - N + 1) / 365
110  LET P = 1 - PN
120  PRINT N, PN, P
130  NEXT N
```

Another potential project problem is the well-known "Locker Doors" problem which is described in the Alberta Education monograph Problem Solving: Challenge for Mathematics (pp. 11-12). Some interesting elaborations and extensions are given in the chapter by J. E. Riley in Problem Solving in the Mathematics Classroom, (MCATA Monograph No. 7; Sid Rachlin, Editor; pp. 95-100).

A problem which could be suitable for less academically inclined students is the one on the right.

Picture this scene from your favourite "Cops and Robbers" television show. A bank robber walks into a bank, taking all the bank workers as hostages. He demands that a suitcase be filled with 1 million dollars in unmarked one dollar bills or he will kill all hostages. His demands are met and he casually fills the suitcase and runs out of the bank with the police in hot pursuit.

Assume that the robber had a suitcase that measured 60 cm by 45 cm by 20 cm.

1. Is it possible for him to get the million dollars in one dollar bills into the suitcase? How high a stack of bills could be used to approximate the thickness of 1 bill? How many suitcases would he need?

2. What would be the weight of the bills with which the robber ran out the door? Could he carry them?

3. If the robber had asked for twenty dollar bills instead of one dollar bills, how many suitcases would he need?

4. If the robber had asked for twenty dollar bills instead of one dollar bills, would he be able to carry the bills?

Conclusion

As the examples given in this section show, many problems have a richness that can be exploited and explored by searching for alternative solution approaches and by posing and seeking to answer spin-off problems. In these ways, the Looking Back stage of problem solving can be used to enrich and enliven the whole problem solving process.

COMPUTER PROBLEMS - LEARNING MATHEMATICS THROUGH COMPUTER PROGRAMMING

Many senior high school students have computer programming skills that they can use profitably in exploring mathematical topics. To do mathematical tasks in a computer context, they have to think carefully about how their experiences in learning mathematics relate to the computer programming environment. The purpose of the problem solving activities discussed in this section is to deepen students' understanding of mathematics, not to develop their programming ability. However, growth in programming skill may be a welcome side-effect.

The value of these activities as mathematics learning experiences derives from the principle that people never know any subject so thoroughly as when they prepare to or attempt to teach it to someone else. Programming a computer may be thought of as "teaching the computer" how to do something. Writing a program demands that students think carefully about the task and determine what instructions and steps they themselves would follow in carrying it out. Then they must translate these operations into language that the computer "understands". An important outcome may be that students begin to recognize important similarities and differences between natural language, mathematical language, and programming languages.

In designing or selecting the mathematical tasks to be carried out by a computer program, teachers should have in mind criteria that will ensure that the experiences are mathematically worthwhile ones for the students. The programs which are given as examples or which are to be written by students should be short, clear, and easy to follow. Teachers should try to minimize the "programming overhead"; that is, the amount of time that students must spend learning about computers and about programming languages before they can even begin to solve any worthwhile problems in mathematics. Only a few of the most fundamental primitives and the most elementary programming techniques should be

introduced. One rule of thumb that might be adopted is that it should be possible to display the entire program on the screen at one time. "Bells and whistles" such as elaborate graphic displays, music, and complicated output formatting should be avoided.

A natural and efficient pedagogical technique in teaching the programming abilities necessary for solving mathematical problems in a computer environment is to introduce programming techniques and syntax by example. It is sometimes best to show students how a key line of code serves some important function, and then get them to use the same idea in a similar program. For example, if the teacher introduces the statement (in BASIC)

```
IF A / B = INT (A / B) THEN PRINT B
"DIVIDES" A
```

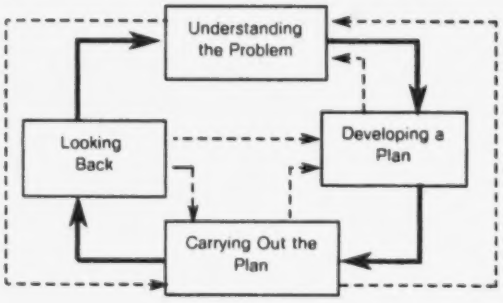
and explains how it can be used to test whether one number is a factor of another, then students can be asked to write programs that list all the factors of a number; that determine whether a number is prime; that find the common factors of two numbers; and so on. A similar technique is for the teacher to provide a program that performs one task and ask students to modify that program so that it performs a related task. For example, students might be given a program which adds two vectors and asked to change it into a program that multiplies a vector by a scalar.

The choice of a particular computer language in which to write will depend upon a number of factors, including the students' familiarity with different languages, the availability and ease of use of the languages, and the relative suitability of the languages for the task to be undertaken. Experience using Logo and BASIC (two widely available languages) suggests that it is feasible to use either of these languages for many tasks, but each language has certain advantages and disadvantages, depending in part upon the task to be performed. For example, Logo programs tend to be longer but clearer because their variable names, procedure names, and modular structures are more like problem solvers' natural thought processes. BASIC programs tend to be shorter, but more like algebraic language than

natural language, and harder to follow without documentation in the form of remark (REM) statements. It is usually easier to use Logo to make graphic displays and to tackle geometric problems, but BASIC probably has the advantage if subscripted variables or arrays are required.

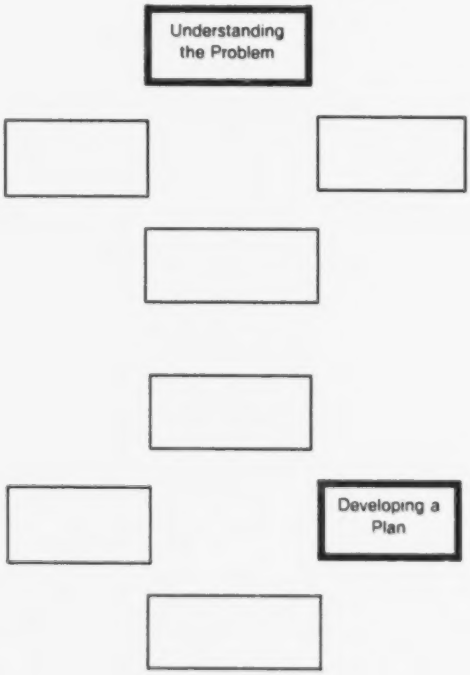
PROBLEM SOLVING MODEL APPLIED TO PROGRAMMING PROBLEMS

The model of the problem solving process we have used is shown at the right. The activities involved in each step are discussed.



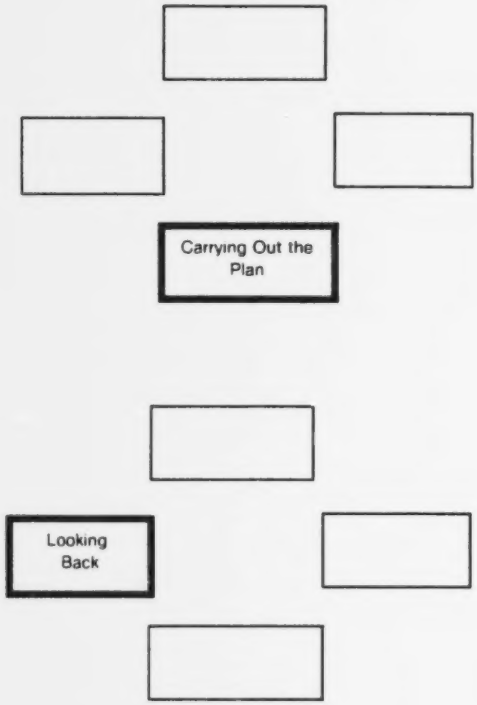
1. **Understanding the Problem**. Usually the task we want the computer to do is one that is familiar to students because they have performed the task themselves many times. Still, the exact nature of the task has to be specified. For example, if we want to write a computer program that adds fractions, we must consider what form the input will take (two numerators and two denominators?) and what form the output should take (should it be in lowest terms? If greater than one, should it be an improper fraction or a mixed number?).

2. **Developing a Plan**. There are many techniques that may be followed when planning a program. Some experts advise outlining the tasks to be done in natural (English) language, then transforming this outline into "pseudo-code" and eventually into program statements in the programming language chosen. Others recommend flow charting; still others suggest writing a master procedure that calls on sub-procedures to be developed later. Sometimes it helps to start by listing and naming the variables that will be used in the program. Whatever the exact nature of the process, it will surely have a number of steps, and it will certainly involve making a general plan of some sort and specifying and refining the plan as the work proceeds.



3. **Carrying Out the Plan.** In this stage, the programmer/problem solver moves from scribbled notes, lists, or flow charts to actual lines of code. Depending upon the approach adopted, various sub-procedures or program segments will have to be entered and tested, perhaps with "dummy data". When bugs are encountered, the programmer may find that an error has occurred in the "carrying out" stage (e.g., a keyboarding error in entering the code) or may decide that the plan needs to be reconsidered (e.g., because a sub-procedure does not do what needs to be done).

4. **Looking Back.** When the problem seems to be solved, looking back is in order. Does the program always work? Are there any special cases of input data that the program cannot handle? Is the program efficient? In loops and iterative processes, are all the iterations necessary? Could the problem be solved in an entirely different way? Could the problem be solved more simply? More elegantly? In answering these and similar questions, it may be useful for students to explain their solutions to others and to discuss the similarities and differences, advantages and disadvantages.



Examples of Programming Problems

Example 1: Operations on Fractions and Operations on Rational Expressions

Background: Since the same general plan that is required to compute

$$\frac{2}{5}+\frac{3}{8}=\frac{16}{40}+\frac{15}{40}=\frac{31}{40}$$

is also used to calculate

$$\frac{x}{x+1}+\frac{2x}{x-3}=$$

$$\frac{x(x-3)+2x(x+1)}{(x+1)(x-3)}$$

teachers often find it worthwhile to review operations with fractional numbers when students are about to begin their work with

rational expressions. This review needs to be done at a high level of generality and should focus on processes such as finding a common denominator, writing the addends as equivalent rational expressions with the common denominator, and recognizing that the sum will be a fraction which has the sum of addends' numerators as its numerator.

Problem 1: Write a program which will calculate the sum of two fractions.

Solution 1: The BASIC program at right solves this problem.

```
10    INPUT "N1 ="; N1
20    INPUT "D1 ="; D1
30    INPUT "N2 ="; N2
40    INPUT "D2 ="; D2
50    NS = D2 * N1 + D1 * N2
60    DS = D1 * D2
70    PRINT
80    PRINT N1; "/"; D1; " +"; N2; "/"; D2;
      " ="; NS; "/"; DS
```

Comments: Students who have low levels of skill with programming might be given this solution, so that they can observe how the syntax of lines 50 and 60 is used to reflect the generalized process of adding two fractions.

Extensions: Having found their own solution to this problem, or having understood how the solution given at the right works, students should have little difficulty modifying the program to make other programs that find the difference, the product, and the quotient of two fractions.

There are no numeric inputs to this program that will cause it to encounter an error, but if a person inputs zero as the value of either of the denominators, the sum will have a denominator of zero, and the program will not indicate that such fractions are not allowed. One extension of the problem might be to add routines that check for fractions with zero denominators and output appropriate error messages.

The program stated above gives its output as a fraction whose denominator is the product of the denominators of the two addends, so this fraction may not be in lowest terms. Another extension would be to modify the program so that the output is in lowest terms. This might be attempted by having the program find the least common multiple of the two addends' denominators, but even in this case, the sum might still not be in lowest terms (e.g., 1/3 + 1/6 = 3/6). Probably the simplest way to ensure that the output is in lowest terms is to call a subroutine that checks for common factors of the numerator and denominator of

the sum. For example, adding the lines at the right to the program above will suffice.

```
75    GOSUB 100
90    END
100   FOR I = NS TO 1 STEP -1
110   IF INT (NS/I) = NS/I AND INT (DS/I) =
      DS/I THEN NS = NS/I: DS = DS/I:
      RETURN
120   NEXT I
130   RETURN
```

Another interesting extension would be to have the program check whether the answer was an improper fraction and, if so, convert it to mixed number form.

Problem 2: Write a program that gives the sum of two rational algebraic expressions.

Discussion: In solving this problem by modifying the previous program, the most important step will be to change the numeric variables to string variables and to change the numeric operations to corresponding string manipulations. Perhaps the most difficult aspect of this problem is the formatting of the output, since the output expression might not fit in the width of the screen, and since the lengths of the various strings cannot be determined in advance, and working with variable-length strings is complicated. One way to solve the problem of output formatting should be to arrange the addends and the sum vertically, and this would be a helpful hint to give to students.

```
10    INPUT "N1 ="; N1$
20    INPUT "D1 ="; D1$
30    INPUT "N2 ="; N2$
40    INPUT "D2 = "; D2$
50    NS$ = "(" + N1$ + ") (" + D2$ + ") +
      (" + N2$ + ") (" + D1$ + ")"
60    DS$ = "(" + D1$ + ") (" + D2$ + ")"
70    PRINT : PRINT
80    PRINT N1$ : PRINT "———":
      PRINT D1$
90    PRINT : PRINT " + " : PRINT
100   PRINT N2$ : PRINT "———":
      PRINT D2$
110   PRINT : PRINT " = " : PRINT
120   PRINT NS$ : PRINT "———":
      PRINT DS$
```

Solution: The program at the right solves the problem, but its output may look a little strange, as the sample run below it shows.

```
]RUN
N1 = A
D1 = X + 3
N2 = B
D2 = Y - 3

  A
———
X+3
 +
  B
———
Y-3
 =
(A)(Y-3) + (B)(X+3)
———————————
   (X+3)(Y-3)
```

Extension: Any solution like this one can be extended quite easily to the case of subtraction, multiplication, and division of rational expressions. Extensions which attempt to multiply out the parts of the expression that are given in factored form here (e.g., routines to change $(x+3)\ (y-3)$ into $xy+3y-3x-9$) are quite difficult to write. But actually this is unnecessary, since the point of the exercise is to reinforce the general procedures.

Example 2: Graphs of Trigonometric Functions

Background: The most important facts about the trigonometric functions include the maxima, minima, and x-intercepts of the functions, their periodicity, and the relations among the functions, such as $\cos(x) = \sin(x+90)$, $\tan(x) = \sin(x) / \cos(x)$, and so on. These facts are usually discussed in terms of the graphs of the trigonometric functions, and

this context is also used to discuss how a simple function like f(x) = sin (x) is related to the more general form f(x) = a sin (bx + c), since the coefficients a, b, and c determine the maxima, minima, phase, and period of this function, but not its basic shape. Problems in which students write computer programs to draw graphs of trigonometric functions require them to have a visual understanding of the effects of the coefficients a, b, and c in the function f (x) = a sin (bx + c), because they must work with these coefficients in order to graph even a simple function like f(x) = sin (x).

Problem 1: Write a program which draws the graph of f(x) = sin (x).

Discussion: The language chosen to solve this problem is Logo, because of its facilities for dealing with graphic displays. High resolution graphics in BASIC could also be used, but the "programming overhead" is probably greater in BASIC than in Logo. All of the procedures given here are written in the Terrapin version of Logo; other versions of Logo may require minor changes in syntax.

Solution: The graphics screen in Logo on an Apple II+/IIe/IIc microcomputer has the point (0,0) at its centre and extends 120 units vertically in both directions and 140 units horizontally in both directions. To draw the coordinate axes, the procedure AXES can be used.

To draw the graph of the function f(x) = sin (x) a recursive procedure (called GRAPH1 in the example at the right) is used. This procedure takes inputs which specify the minimum and maximum x values to be plotted. The primitive SETXY moves the turtle to the screen position specified by the two numeric values which follow SETXY, and as the turtle moves from point to point it draws the line connecting them. The expressions ":XMIN" and ":XMAX" refer to the values of the two local variables "XMIN" and "XMAX". The next to last line "GRAPH1 (:XMIN + 1) :XMAX" calls for the procedure GRAPH1 to execute itself again, with a value for XMIN that is one more than its previous value. The first line of the procedure indicates that the recursion is to continue until the value

```
TO AXES
   HOME
   FD 110  ARROW  BK 110  RT 90
   FD 130  ARROW  BK 130  RT 90
   FD 110  ARROW  BK 110  RT 90
   FD 130  ARROW  BK 130  RT 90
END

TO ARROW
   RT 135  FD 5  BK 5  LT 135
   LT 135  FD 5  BK 5  RT 135

TO GRAPH1 :XMIN :XMAX
   IF :XMIN > :XMAX STOP
   SETXY :XMIN 100 * SIN ( :XMIN)
   GRAPH1 ( :XMIN + 1) :XMAX
END
```

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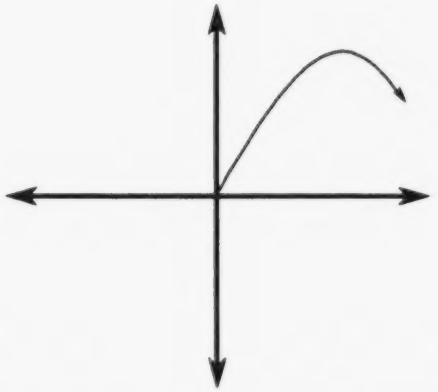
of XMIN becomes greater than the originally specified (and never changed) value of XMAX.

The built-in function SIN ( ) takes its input in degrees and outputs the sine of that angle. Since the width of the screen is 140 turtle steps each side of the centre, we can easily graph the sine of angles up to this value. Since the output of SIN ( ) ranges from -1 to +1 but the screen extends 120 turtle steps above and below the centre, the procedure uses 100 times the sine as the value for the y-coordinate. When the commands DRAW AXES GRAPH1 0 135 are given, the first graph at the right is produced.

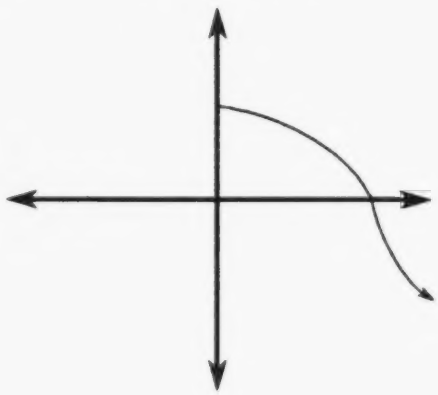


Problem 2: Draw the graph of the function f(x) = cos (x).

Solution: The procedure GRAPH2 is a slight modification of GRAPH1. The commands DRAW AXES GRAPH2 0 135 were used to produce the solution shown at the right.

```
TO GRAPH2 :XMIN :XMAX
    IF :XMIN > :XMAX STOP
    SETXY :XMIN 100 * COS ( :XMIN )
    GRAPH2 ( :XMIN + 1 ) :XMAX
END
```

Discussion: This procedure can also be used to draw the graph of f (x) = cos (x) for x from -90 to +90 degrees. But students will find a bug in this procedure if the first input of GRAPH2 (the minimum x-value) is not 0 or an angle whose cosine is 0. When the turtle's pen is down and its initial position is different from its position the first time that the SETXY command is executed, a line will be drawn from the turtle's starting position to the first point plotted. This problem can be overcome by lifting the pen (PU), positioning the turtle to begin drawing the graph, then setting the pen down (PD), as the following example shows.



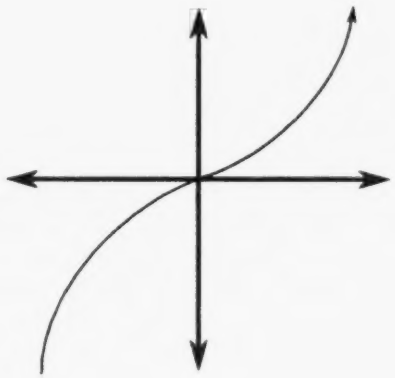
Problem 3: Graph the function f (x) = tan (x).

Discussion: This problem is a little more difficult because the tangent function is not built into Logo. In computing tan (x) from sin (x) / cos (x) it is important to use brackets carefully so that this expression is interpreted as (sin (x)) / (cos (x)), not as sin (x /cos (x)).

The graph of the tangent function has asymptotes at every positive and negative multiple of 90 degrees, so we will graph the function for values of x from -85 to +85 degrees. Later, if we wish to extend this problem to include x values which are asymptotes of the function, we will have to find ways to prevent the graph from going off the screen and ways to avoid division by zero error messages.

Solution: The procedure GRAPH3 shown at the right can be used to solve this problem. The commands DRAW AXES PU SETXY -85 (10 • (SIN (-85)) / (COS (-85)) PD GRAPH3 -85 85 will draw the graph starting with x at -85 degrees. The factor 10 in the line SETXY :XMIN 10 • (SIN (:XMIN)) / COS (:XMIN) was chosen to give the graph a reasonable vertical scale in this horizontal interval.

```
TO GRAPH3 :XMIN :XMAX
   IF :XMIN > :XMAX STOP
   SETXY :XMIN 10 • (SIN ( :XMIN) ) /
COS ( :XMIN)
   GRAPH3 ( :XMIN + 1 ) :XMAX
END
```



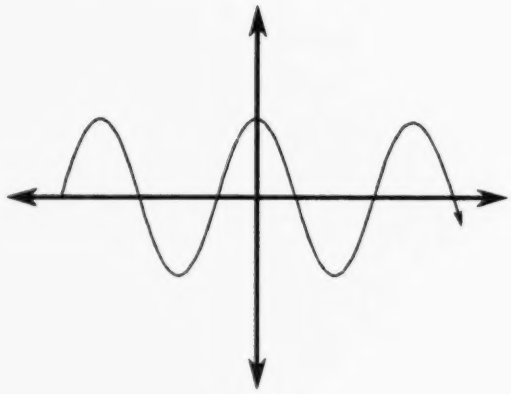
Problem 4: Draw the graph of f(x) = cos (x) for x from -450 to +540 degrees.

Discussion: This problem is difficult because the domain of the graph is larger than the width of the screen. If we were to use a procedure like the previous ones, the graph would run off the screen, or "wrap around", or cause an error message to be given. In order to fit the graph of 540 degrees into a space of only 140 turtle steps, we need to apply a scaling process that represents more than one degree in each turtle step. A scale factor of 0.2 turtle steps for each degree is suitable.

The solution: The procedure GRAPH4 given at the right can be used to solve the problem; the commands DRAW AXES GRAPH4 -450 540 produce the graph shown. Note that although the variable XMIN ranges from -450 to +540, the x-coordinates plotted are only 0.2 times as large, ranging from -90 to +108 turtle steps. In this procedure the maximum y–coordinate is only 50, because the second input to SETXY is now 50 • COS (:XMIN).

```
TO GRAPH4 :XMIN :XMAX
   IF :XMIN > :XMAX STOP
   SETXY 0.2 • :XMIN 50 • COS ( :XMIN )
   GRAPH4 ( :XMIN + 1 ) :XMAX
END
```

Additional Problems: From the examples given above it can be seen that similar Logo procedures can be used to draw the graphs of other trigonometric functions over different intervals with various modifications to the amplitude, period, and phase. The thinking that a programmer must do to solve such programming problems is very similar (but not quite identical) to the thinking that needs to be done to determine the effects of the coefficients in the functions we discussed earlier. For example, in f (x) = 5 sin (2x + 90) we note that the amplitude is 5 (just as we adjusted the vertical scale factor in previous problems), we find that the period is 180 degrees (half of 360 degrees because "2x goes twice as fast as x"), and we see that the phase is shifted 90 degrees (since we are finding the sine of an angle that is 90 more than 2x). The ability to think in this way about complicated trigonometric functions is a mark of thorough understanding of them.



ADDITIONAL EXAMPLES

There are many other topics in senior high school mathematics where writing computer programs to perform mathematical tasks can help to deepen students' understanding of that mathematics. For example, in number theory we could start with a routine that determines whether one number is a factor of another and use it to build a program that indicates whether a counting number is prime or composite, a program to find the LCM or GCF of two numbers, a program which tells whether a number is perfect, abundant or deficient, and many others.

We could also write a program that takes values of the coefficients of the equation

$$ax^2 + bx + c = 0$$

and uses the quadratic formula to output the values of x that satisfy it. More complicated versions of the program could be written to handle situations in which the roots of the equation are complex numbers.

Such activities can have significant value if they stimulate students to think in general terms about mathematics as they translate that mathematics into computer language contexts.

EXAMPLE PROBLEMS

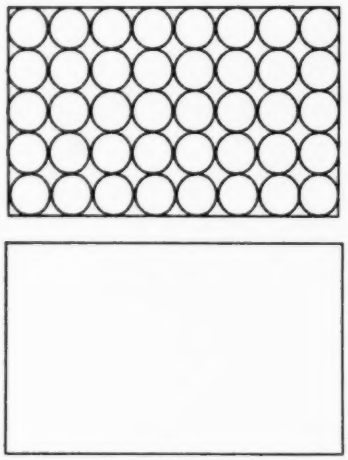
The following are illustrative of problems which can be successfully given to high school students. They are given with some of the ways of solution which could be expected, depending on the background of the student.

The importance of the looking back stage mentioned in the previous pages is noted again within this section. Each problem has a number of extensions listed, some of which parallel the original problem, some which take the same concepts and apply them in a different way. Students should always be encouraged to extend their problem solutions to new situations by creating new problems or by solving already posed problems using their newly acquired knowledge.

ONE UP

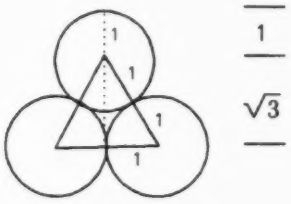
A soft drink manufacturer delivers its products in cases holding exactly 40 bottles packed side by side. A note in the company suggestion box pointed out that the firm could pack 41 bottles in the same cases by simply rearranging the bottles.

The diagram on the right shows a full case of 40 bottles. The figure below it is an empty case. Show how 41 bottles can be arranged in the empty case. (Higgins, A.M. Geometry Problems. Portland, ME: J. Weston Walch, 1971.)

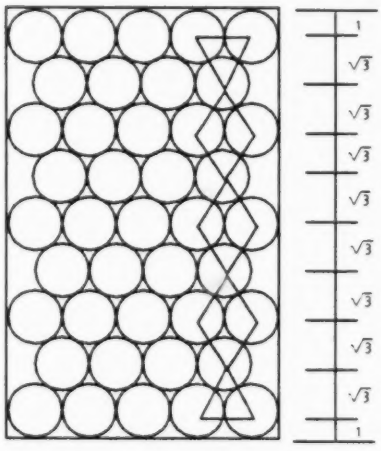


SOLUTION #1

Prerequisite Knowledge: Pythagorean Theorem, Tangent Circles, Radicals
Strand: Geometry
Strategy: Guess and check.



Some students may simply guess at various arrangements that might allow 41 bottles to fit in the case, eventually coming up with a 5, 4, 5, 4, 5, 4, 5, 4, 5 arrangement. This however, is only a guess and will have to be checked to see if the length of the case of bottles in this arrangement is longer than 16 units if it is assumed that the radius of each bottle is one unit. A triangle must be drawn as shown and, using the Pythagorean Theorem, the distance between consecutive centres will be $\sqrt{3}$. There are eight of these distances plus the two radii of 1 unit each at the top and bottom. The total length of the case would be $2 + 8\sqrt{3}$ or approximately 15.86 which is less than the 16 units in the original arrangement.



A second common guess that students make is looking at the case sideways, and making rows of 8, 7, 8, 7, 8, 7. This arrangement holds 45 bottles but is wider than 10 units when checked as above.

SOLUTION #2

Prerequisite Knowledge: Ability to model/draw Pythagorean Theorem, Tangent Circles, Radicals
Strand: Geometry
Strategy: Make a model/draw a diagram.

A diagram can be carefully drawn, but it is probably easier to attempt the problem by modeling. A good simple model of the problem can be made using pennies. The students can measure the length of 8 pennies and 5 pennies to determine the dimensions of the original case. With these dimensions, they can make a "case" with masking tape on their desktop. Manipulating 41 pennies will give the desired arrangement. Again, even though it appears the pennies in the 5,4,5,4,5,4,5,4,5 arrangement will fit, the question should be checked as was done in solution #1.

Extending the Problem

1. Are there other size cases holding less than 40 bottles which could hold an extra bottle?

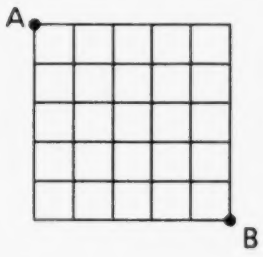
2. What would be the smallest size case which would hold 2 extra bottles?

3. Can you generalize your solution for 1 extra, 2 extra, 3 extra, ...*n* extra bottles added to the case?

4. Make a 3-dimensional case holding bowling balls or grapefruits. Is it possible to get more in the case in an "irregular" arrangement or a "regular" arrangement? What are the smallest dimensions that allow for the addition of at least one ball or grapefruit?

THE PATH PROBLEM

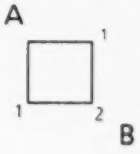
How many different paths are there from the Point A to the Point B on a 5 x 5 square if only moves to the right or downward along the sides of the squares are permitted?



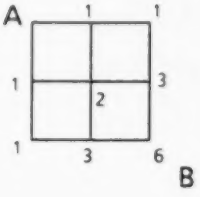
SOLUTION #1

Prerequisite Knowledge: Counting skills, Pascal's triangle
Strand: Discrete Mathematics
Strategy: Solve a simpler problem, look for a pattern.

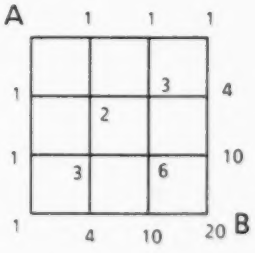
For a 1 x 1 square the number of paths from A to B is 2. In addition note the number of paths from A to each of the other corners of the square.



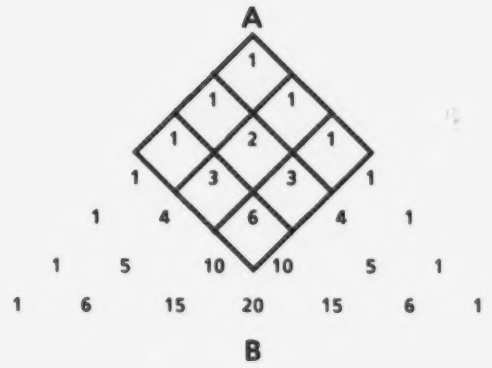
For a 2 x 2 square, the number of paths from A to B is 6. Once again, note the number of paths from A to each of the intersections in the grid.



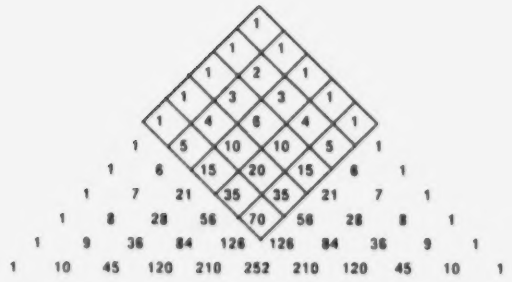
For a 3 x 3 square, the number of paths from A to B is 20. As in the previous cases, note the number of paths from A to each other intersection in the grid.



It should now be apparent that the numbers being generated occur in Pascal's triangle.



In order to determine the number of paths in a 5×5 square, simply extend the triangle.



Hence, the number of paths from A to B on a 5×5 square is 252.

In addition to determining the number of paths from A to B, Pascal's triangle can also be used to determine the number of paths from A to any other point on the 5×5 square.

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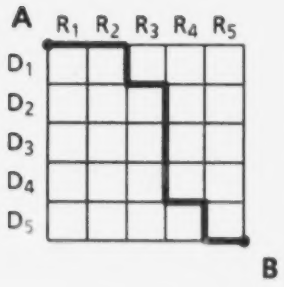
SOLUTION #2

Prerequisite Knowledge: Permutations
Strand: Discrete Mathematics
Strategy: Change your point of view.

A path from A to B will consist of some permutation of the symbols R_1, R_2, R_3, R_4, R_5, D_1, D_2, D_3, D_4, D_5 where

a. R_i represents the i^{th} move to the right
b. D_i represents the i^{th} move downward

One such path would be R_1, R_2, D_1, R_3, D_2, D_3, D_4, R_4, D_5, R_5 (see diagram).



Now the number of different permutations of a 10 element set is 10! However, in some of these permutations, R_2 will precede R_1, etc. In order to eliminate these permutations, treat the R_is as being indistinguishable, and similarly for the D_is. Then the number of paths from A to B is $\frac{10!}{5!\,5!} = 252$.

Extending the Problem

1. How many paths are there between opposite corners of
   a. a 6 x 6 square?
   b. a 7 x 7 square?
   c. a N x N square?

   (Answers: a. 924, b. 3432, c. ${}_{2N}C_N$)

2. Downtown Edmonton is laid out on a rectangular grid. If only moves eastward or northward are permitted, how many different paths are there from the corner of 109 Street and 101 Avenue to the corner of 103 Street and 104 Avenue?

   (Answer: 84)

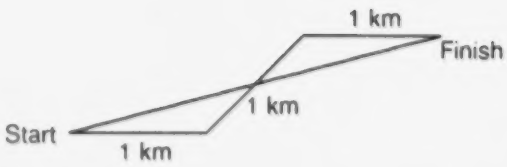
WALK STRAIGHT!

Suppose a person walks one kilometre east, then one kilometre northeast, then another kilometre east. What is the distance between the person's initial and final positions?

SOLUTION #1

Prerequisite Knowledge: Measurement with a ruler and protractor, knowledge of vector addition, construction techniques with compass and straight edge
Strand: Geometry
Strategy: Construct a scale diagram.

If we choose some convenient measure (e.g., 2 cm = 1 km), it is possible to draw a diagram to represent the problem. Care must be taken to construct accurately the line representing "1 km northeast".

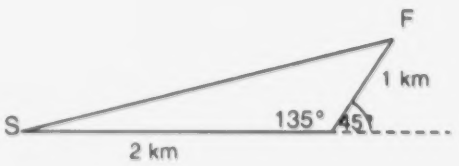


The distance between the start and the finish is about 2.8 km. (Greater accuracy will be achieved with larger diagrams.)

SOLUTION #2

Prerequisite Knowledge: Solving oblique triangles
Strand: Trigonometry
Strategy: Solve a simpler problem, law of cosines.

The same final position will be achieved if the person had initially walked 2 km east and then 1 km northeast.



Using the cosine law

$$SF^2 = 2^2 + 1^2 - 2(2)(1)\cos 135°$$
$$SF^2 \doteq 7.9$$
$$SF \doteq 2.8 \text{ km}$$

Note: Since $\cos 135° = \frac{-\sqrt{2}}{2}$ we could say

$$SF = \sqrt{(5+2\sqrt{2})} \text{ km}$$

SOLUTION #3

Prerequisite Knowledge: Vector operations
Strand: Trigonometry and Vectors
Strategy: Vector addition.

The three parts of the walk can be designated $\vec{V}_1$ (1 km E), $\vec{V}_2$ (1 km NE), and $\vec{V}_3$ (1 km E).

$\vec{V}_1 = (1, 0°)$
$\vec{V}_2 = (1, 45°)$
$\vec{V}_3 = (1, 0°)$

In rectangular notation, they would be

$\vec{V}_1 = [1, 0]$
$\vec{V}_2 = [.71, .71]$
$\vec{V}_3 = [1, 0]$

The resultant, $\vec{R}$, is the sum of $\vec{V}1$, $\vec{V}2$ and $\vec{V}3$.

$\vec{R} = \vec{V}_1 + \vec{V}_2 + \vec{V}_3$
$\vec{R} = (1, 0) + (.71, .71) + (1, 0)$
$\vec{R} = (2.71, .71)$

To find the distance walked, we need $|\vec{R}|$.

$|\vec{R}| = \sqrt{[(2.71)^2 + (.71)^2]}$
$|\vec{R}| \doteq 2.8$

The distance walked is 2.8 km.

Extending the Problem

1. A person walked 1 km east, then 2 km north, then 3 km west, then 4 km south. How far was this person from the starting point? In what direction?

2. If he or she continued on in the above fashion, how far would he or she be from the starting point after the 10 km leg? The n km leg?

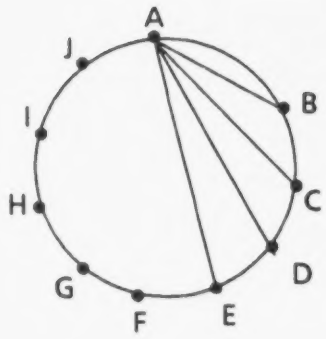
THE HANDSHAKE PROBLEM

Ten delegates from ten countries meet at a conference on world peace. If each delegate shakes hands with the delegate from each of the other countries, how many handshakes would take place?

SOLUTION #1

Prerequisite Knowledge: Counting skills
Strand: Number Theory
Strategy: Act it out or make a diagram and count the possibilities.

If each of the delegates is represented by a letter of the alphabet (or by individual students), then the number of handshakes can be determined by counting the number of ways that the letters or students can be paired off. One way of accomplishing this would be to count the number of lines that can be drawn joining the letters displayed around a circle.



Hence, the number of handshakes is 45.

SOLUTION #2

Prerequisite Knowledge: Arithmetic skills
Strand: Number Theory
Strategy: Use logic.

Since each of the 10 delegates must shake hands with 9 other delegates, it would appear that the total number of handshakes is $10 \times 9 = 90$. However, in counting this way, each handshake will be counted twice. Hence, the number of handshakes is

$$\frac{10 \times 9}{2} = 45.$$

SOLUTION #3

Prerequisite Knowledge: Arithmetic skills
Strand: Number Theory
Strategy: Solve a simpler problem, make an organized list, look for a pattern.

Number of People	Number of Handshakes
2	1
3	3 = 1 + 2
4	6 = 1 + 2 + 3
5	10 = 1 + 2 + 3 + 4
6	15 = 1 + 2 + 3 + 4 + 5
.	.
.	.
.	.

Hence, for 10 people, the number of handshakes is 1 + 2 + 3 + ... + 9 = 45.

SOLUTION #4

Prerequisite Knowledge: Combinations
Strand: Discrete Mathematics
Strategy: Change your point of view, and use a formula.

Since each combination of 2 delegates represents a handshake, the problem can be rephrased "how many 2-element combinations can be formed in a set of 10 elements?" Then the number of handshakes is

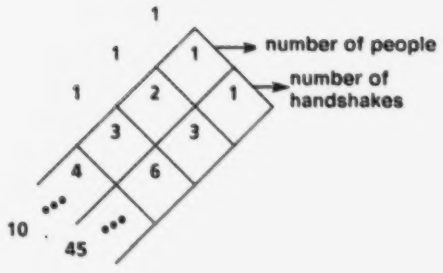
$$_{10}C_2 = \frac{10 \times 9}{2} = 45$$

SOLUTION #5

Prerequisite Knowledge: Pascal's triangle
Strand: Discrete Mathematics
Strategy: Solve a simpler problem, make an organized list, and look for a pattern.

In solution #3, a problem solver may recognize that the 2 sequences of numbers found in the table occur in Pascal's triangle.

By extending Pascal's triangle, the number of handshakes among 10 people is found to be 45.



Extending the Problem

1. Using any method, how many handshakes would take place for

   a. 12 people?
   b. 15 people?
   c. 20 people?

   (Answers: a. 66, b. 105, c. 190)

2. In a roundrobin ping-pong tournament, each player must play each of the other players once. How many games must be scheduled in a tournament involving 25 players? (Answer: 300)

3. If N countries are represented at the peace conference, then derive an expression for the number of handshakes that will take place among the delegates.

   (Answer:
   $\frac{N(N-1)}{2}$, $1+2+3+\ldots+(N-1)$, ${}_NC_2$)

4. At one conference, 120 handshakes were exchanged among the delegates. How many delegates attended the conference?

   (Answer: 16)

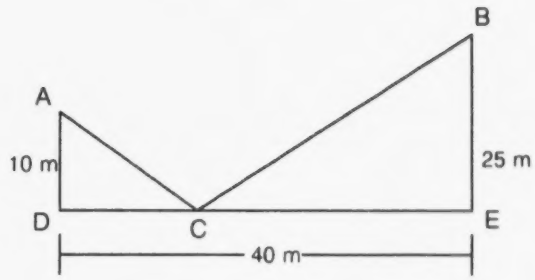
THE TELEPHONE POLE PROBLEM

Two telephone poles are to be placed 40 m apart. The poles must be anchored by guy wires to a peg located somewhere between them. Where should the peg be placed so that the sum of the lengths of the guy wires is minimum?

SOLUTION #1

Prerequisite Knowledge: Pythagorean Theorem
Strand: Geometry
Strategy: Guess and test, diagram.

We draw a diagram and label it, with the peg at C. We want to find out where C should be so that AC + BC will be as short as possible.



Let us set up a chart and agree to measure the position of the peg from the base of the taller pole (CE). Using the Pythagorean Theorem, we can find the lengths of AC and BC.

Suppose we place the peg in the middle between the two poles, i.e., CE = 20.

To find AC

$AC^2 = AD^2 + DC^2$
$AC^2 = 10^2 + 20^2$
$AC \doteq 22.36$

To find BC

$BC^2 = BE^2 + CE^2$
$BC^2 = 25^2 + 20^2$
$BC \doteq 32.02$

We use these values to fill in the following chart and continue to fill in the chart for different values of CE until we find the minimum sum.

CE	AC	BC	AC + BC
20	22.4	32.0	54.4
15	26.9	29.2	56.1
25	18.0	35.4	53.4
28	15.6	37.5	53.1
29	15.6	37.5	53.1

Since the sum is the same for CE = 28 or CE = 29, it seems reasonable to try CE = 28.5 m.

It appears that the sum will be about the same if the peg is placed anywhere between 28 and 29 metres from the taller pole, and the minimum amount of wire needed is 53.1 m.

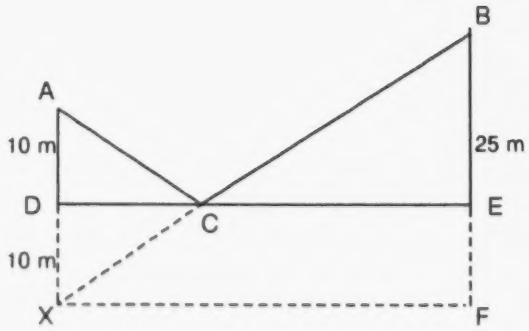
(Since this is not strictly true, the teacher may encourage a student to search for AC + BC using more accurate values.)

CE	AC	BC	AC + BC
28.5	15.2	37.9	53.1

SOLUTION #2

Prerequisite Knowledge: Reflections, similar triangles
Strand: Geometry, Trigonometry
Strategy: Solve a simpler problem, diagram.

If the 10 m pole is reflected through the 40 m line, XC = AC and BCX would equal the minimum sum of AC + BC.



We want to find the length of CE. We know that ΔBCE and ΔBXF are similar with XF = 40 and BF = 35.

$$\frac{CE}{BE} = \frac{XF}{BF}$$

$$\frac{CE}{25} = \frac{40}{35}$$

$$CE = \frac{25\ X\ 40}{35}$$

$$CE = 28\frac{4}{7}\ or\ approximately\ 28.6$$

The peg should be placed 28.6 m from the taller pole for the length of the guy wires to be a minimum. The length of the wire will be 53.2 m.

SOLUTION # 3

Prerequisite Knowledge: Functions and elementary differential calculus
Strand: Calculus
Strategy: Diagram, create a function to describe length of wires, differentiation.

Again we can draw the diagram and label it as shown.

The length of each of the guy wires can be expressed as a function.

$AC = [(40 - x)^2 + 100]^{\frac{1}{2}}$

$BC = (x^2 + 625)^{\frac{1}{2}}$

The total length of wire needed (w) is a function of the distance, x, from the 25 m pole.

$w = BC + AC$

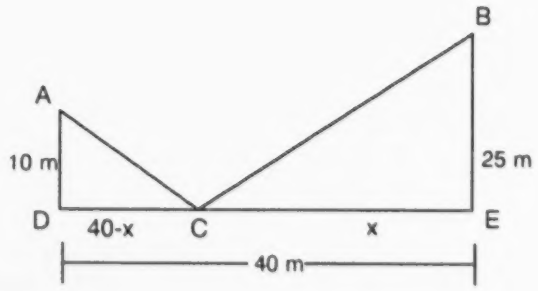
$w(x) = (x^2 + 625)^{\frac{1}{2}} + [(40 - x)^2 + 100]^{\frac{1}{2}}$

$$\frac{dw}{dx} = \frac{x}{(x^2+625)^{\frac{1}{2}}} - \frac{(40-x)}{[(40-x)^2+100]^{\frac{1}{2}}}$$

To find the minimum, we find the value of x for which $\frac{dw}{dx} = 0$

$$x = 28\frac{4}{7}$$

The peg anchoring the guy wire should be placed $28\frac{4}{7}$ m from the 25 m pole.



Extending the Problem

1. A man standing on the bank of a wide canyon sees a rock directly across from him on the opposite bank. How can he determine the distance to the rock if he cannot cross the canyon, and has only a very long measuring tape?

2. The lengths of two sides of a triangle are given. How must they be placed so that the area of the triangle will be a maximum?

3. The length, width and height of a room are 7 m, 6 m and 3 m respectively. A fly sitting in a corner where the roof meets the wall must walk to the corner diagonally across from it, where the floor meets the wall. Find the length of the shortest path the fly can take.

A PROBLEM BANK

The following is a selection of problems loosely organized by strand. They are examples of problems which are appropriate for high school students. Many can be solved in more than one way, and many can be extended to create new problems.

NUMBER THEORY

1. What is the units digit for the following sum?

   $13^{841} + 17^{508} + 24^{617}$

   8

2. Given: $A = \sqrt{x} + \sqrt{(2x-1)} + \sqrt{x} - \sqrt{(2x-1)}$ and x is a real number.

   a. What is the minimum value that can be assigned to x?
   b. For what values of x is $A = \sqrt{2}$?
   c. If A = 2, what is the value of x?

   a. $\frac{1}{2}$
   b. $\frac{1}{2} \leq x \leq 1$
   c. **1.5**

3. a. Which has more zeros at the end of the number: $(10^3)^{100}$ or $(10^3)!$? Why? How many more?

   b. Find all the natural numbers, n such that n! ends in 32 zeros.

   c. Find n!, if n ends in 17 zeros.

   a. $(10^3)^{100}$
   b. **130, 131, 132, 133, 134**
   c. **There is no such natural number**

4. A census taker came to a house where a man lived with his 3 daughters. "What are the ages of your 3 daughters?" asked the census taker. "The product of their ages is 72, and the sum of their ages is my house number" replied the man. "But clearly that is not enough information", insisted the census taker. "All right," said the man, "The oldest in years loves chocolate milk." What are the ages of the 3 daughters?

   3, 3, 8

5. Express $\sqrt{(52 + 30\sqrt{3})}$ in the form $a + b\sqrt{3}$, where $a, b \in Q^+$

   $5 + 3\sqrt{3}$

6. Find the ordered pair of real numbers (x,y) which satisfies $16^x - 16^y = 192$ and $4^x - 4^y = 8$

   2, $\frac{3}{2}$

7. Legally married in Alberta, my neighbour has reached a square age. The product of the digits of his age is his wife's age. The age of their daughter is the sum of the digits of her father's age, and the age of their son is the sum of the digits of his mother's age. How old are they?

   Father 49
   Mother 36
   Daughter 13
   Son 9

8. When you write the numbers 1-100, how many times do you write the digit "9"?

   19

9. Evaluate the radical

$$\left(\frac{1\times2\times4+2\times4\times8+3\times6\times12+\ldots}{1\times3\times9+2\times6\times18+3\times9\times27+\ldots}\right)^{1/3}$$

   $\frac{2}{3}$

10. Find a positive integer which gives a perfect square if 100 is added to it, and another perfect square if 168 is added to it.

    156

11. The positive integers are written down in order starting with 1: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 ...
    What digit is in the 319 468th position?

    1

12. If 3093 digits were used to number the pages of a book, how many pages are in the book?

    1050

67

13. How many positive integers less than 1000 are divisible neither by 5 nor by 7?

686

14. An integer consists of 1985 digits each of which is a 2. What is the remainder when this integer is divided by 9?

1

15. Some numbers may be divided into four parts which have odd properties (here odd is not the opposite of even). For instance, 45 may be split into four parts, such that if one is increased by 2, one is diminished by 2, one is multiplied by 2, and one is divided by 2, the results are the same. Find what the four parts are and what the unique result is for all four operations.

8, 12, 5, 20

16. The owner hired 3 watchmen to guard his orchard, but a thief still got in and stole some apples. On the way out, the thief met each watchman, one at a time. To each he gave $\frac{1}{2}$ of the apples he had then, and 2 more besides. He escaped with one apple. How many did he steal originally?

36 apples

17. The holiday greeting, A MERRY XMAS TO ALL, is a cryptarithm in which each of the letters is the unique representation of a digit in the decimal scale, and each word is a square integer. Find the numerical interpretation if in each word the sum of the digits is also a square.

9 34225 7396 81 900

18. Find the remainder when 3^{1986} is divided by 13.

1

19. If the first fifty natural numbers are multiplied together, how many zeros will appear in the product at the end?

12

20. An "abundant" number is a natural number the sum of whose proper divisors is greater than the number itself. What is the smallest abundant number?

12

21. When eggs in a basket are removed 2, 3, 4, 5, and 6 at a time, 1, 2, 3, 4, and 5 eggs remain, respectively. When they are taken out 7 at a time, none are left over. Find the smallest number of eggs that could be in the basket.

119

22. Take two numbers whose sum is one. Which do you think is larger: the square of the smaller added to the larger or the square of the larger added to the smaller? Why?

They are the same

23. Which natural numbers x, y satisfy the equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{xy}=1?$$

x = 2, y = 3
x = 3, y = 2

24. Find a positive integer t such that $s^2 - t^2$ is a prime number and s = 15.

t = 14

25. Show that $n^2 - 1$, $n \in I$ cannot generate more than one prime integer.

3 is the only value of $n^2 - 1$ which is prime

68

DISCRETE MATHEMATICS

1. Three freighters leave St. John's, Newfoundland for Montreal, Quebec at the same time. One ship takes 20 d to make the round trip, another ship takes 16 d, and the final ship takes 12 d. If these three ships continuously make this round trip between St. John's and Montreal, how long will it be before all three ships are in the same port at the same time again?

   240 days

2. In how many ways can ten dollars be changed into dimes and quarters?

   21

3. To conserve the contents of a 16 mL bottle of tonic, a castaway adopts the following procedure. On the first day he drinks 1 ml of tonic, then refills the bottle with water; on the second day he drinks 2 ml of the mixture, then refills the bottle with water; on the third day he drinks 3 ml of the mixture, and again refills the bottle with water. The procedure is continued for succeeding days until the bottle is empty. How many millilitres of water does he drink? On what day does he drink the most tonic?

   a. **120 ml**
   b. **Day 4**

4. Given that 24, b, c are the first three terms of an arithmetic sequence, with non-zero common difference, and that 24, c, b are the first 3 terms of a geometric sequence, find b and c.

   b. **6,**
   c. **-12**

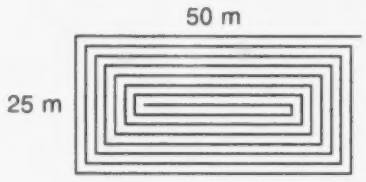
5. Given that the quartic polynomial $x^4 - ax^3 + bx^2 - cx + d$ has one zero of multiplicity 4, show that $a \times d$ is directly proportional to $b \times c$.

   Use Binomial Expansion

6. The middle pearl on a string of 33 pearls is the largest and most expensive. Starting from one end, each pearl is worth $100.00 more than the one before, up to the middle. From the other end, each pearl is worth $150.00 more than the one before up to the middle. The string of pearls is worth $65 000. What is the value of the middle pearl?

   $3000

7. If a person cuts a rectangular lawn that is 25 m by 50 m in a curved pattern with a mower that is 0.5 m wide, how far does that person walk?



   2525 m

8. If $a \Delta b = (a + b) + (ab) + b$, what is $(5 \Delta 7) \Delta 3$?

   222

9. On the first day, Jack's beanstalk increased its height by $\frac{1}{2}$; on the second day by $\frac{1}{3}$; and on the third day by $\frac{1}{4}$; and so on. How long did it take to achieve its maximum height?

   198 days

10. What is the next number in this sequence?
    0, 2, 24, 252, ...

    3120

ALGEBRA

1. Let $x(a-b)^{-1} = y(b-c)^{-1} = z(c-a)^{-1} = 1985$, where a, b, c, x, y, and z are real numbers. What is the value of $x + y + z$?

0

2. Buses travel east along a highway at 40 km/h and they are spaced exactly 36 min apart. A pedestrian is walking east along the highway at 8 km/h. A bus travelling east passes the pedestrian. How many minutes will pass until the next bus, travelling east passes the pedestrian?

45 min

3. Solve for x: $2 = x^{x^{x^{\cdots}}}$

$\sqrt{2}$

4. If $x = \log_a 2^b$, then $\log_b a^x =$

$\frac{1}{2}$

5. Find x if $3^{20} + 3^{20} + 3^{20} = 3^x$.

21

6. Find integers x and y such that

$3^x \times 6^y = 24$

x = -2, y = 3

7. Find the value of k, $k \in R$, such that the two solutions of the equation

$$-(1+k)x^2 + (3k-5)x - 4 = 0$$

differ by 3.

K = 0

8. Find the value of x which satisfies

$\sqrt[3]{(x\sqrt{x})} = 4$

16

9. There is a function machine that operates on a given input number (and no other numbers) using addition, subtraction, multiplication, division, or any combination of these four operations. Complete the following table. What is the function?

INPUT	OUTPUT
1	1
2	9
3	29
4	67
5	129
6	221
7	...
8	...
9	...
10	...

a. **7-334, 8-519, 9-737, 10-1009**
b. **$x^3 + x - 1$**

10. Two mathematicians are swimming laps back and forth across the length of a pool with constant speeds and essentially instantaneous turns at the ends of the pool. Determine the length of the pool if they leave from opposite ends of the pool at the same time and meet for the first time 21 m from one end and continue on their ways to ends of the pool, turn around, and meet again 8 m from the opposite end.

55 m

11. The sum of the squares of five integers in arithmetic progression is 2910 and the sum of their cubes is 17 380. Find the integers and list them from smallest to largest.

-32, -15, 2, 19, 36

12. a. If $(\log_2 3)(\log_3 4)(\log_4 5) = \log_2 x$, find x in simplest form.

b. What is the general pattern of identity used to generate this type of question?

a. **5**
b. **$(\log_n k)(\log_k n^2)(\log_n 2p) = \log_n p$**

13. A father budgeted $60.00 to distribute equally among his children for spending money at the beach. When two young cousins joined the party and shared in the equal distribution, each child received $2.50 less than had been planned. How many children were in the party?

8

14. A girl entered a store and bought x flowers for y dollars (x and y are integers). When she was about to leave, the clerk said, "If you buy 10 more flowers I will give you all for $2.00, and you will save 80¢ a dozen". Find x and y.

x = 5, y = 1

15. In a pet store, the cats sold for $10.00 each and the canaries for $15.00. The total value was $360.00. One night the animals got loose. Two cats disappeared and the others ate half of the canaries. If the total value was now $220.00, how many of each were present before the escape?

12 cats
16 canaries

16. The variables x and y satisfy the simultaneous equations

$$x + ky = 5$$
$$kx + 4y = 8 + k$$

where $k \in R$. Find the values of the real number k for which these equations have

a. no solution
b. an infinite number of solutions.

k = -2
k = 2

17. The numerator of a certain fraction is 3 less than the denominator. If the numerator is tripled and the denominator is increased by 7, the value of the resulting fraction is 3/2. Find the original fraction.

$$\frac{10}{13}$$

18. Find the real value of x such that $x \log_2 3 = \log_{10} 3$.

x = log 2 or .3010

19. Darryl ate 100 peanut butter cups in five days. Each day he ate six more than he ate the previous day. How many peanut butter cups did Darryl eat on the first day?

8

20. a, b, and c are three positive integers such that a^2, $2b^3$, and $3c^5$ are equal. Find a, b, and c.

a = $2^5 \times 3^3$
b = $2^3 \times 3^2$
c = $2^2 \times 3$

21. A book has 152 pages. Every page has an average of 45 lines and each line has about 68 characters. If the book was reprinted with larger lettering resulting in 32 lines per page and 51 characters per line, how many pages would the new text contain?

285

22. A toy store is buying stuffed horses and stuffed cows for a total of $1770. The wholesale price for a horse is $31 and $21 for a cow. How many horses and cows did they buy? (Is there more than one solution?)

9 horses, 71 cows
30 horses, 40 cows
51 horses, 9 cows

GEOMETRY

1. A circular field is to be divided into 4 equal parts using 3 fences. If the fences are of equal length, how will you do it?

   Various answers

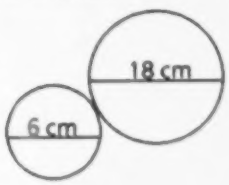
2. A tape is placed snugly around the earth at the equator. If another tape 2 m longer was placed a uniform distance above the first, how far above the ground would it be? (Assume that the earth is a smooth sphere with a circumference of 40 000 km.)

   31.8 cm

3. Each edge of a cubical box has a length of 1 m. The box contains 9 spherical balls with the same radius r. The centre of one ball is at the centre of the cube and it touches the other 8 balls. Each of the other 8 balls touches 3 sides of the box, i.e., the balls are tightly packed in the box. Find r.

   $$\frac{\sqrt{3}}{6}$$

4. Two poles with diameters 6 cm and 18 cm are placed as in the diagram and are bound together with wire. What is the length of the shortest wire that will go around them?



   $(12\sqrt{3} + 14\pi)$cm

5. A regular polygon with n sides is inscribed in a circle with radius r. The area of the polygon is $3r^2$. What is n?

   12

6. The perimeter of a right triangle is 60 cm and the altitude perpendicular to the hypotenuse is 12 cm. Find the three sides of the triangle.

   15, 20, 25

7. Is it possible to bend a piece of wire 10 cm long to form the legs of a right triangle with an area of 15 cm^2?

   No

8. A rectangular room has a height of 4 m. If the length and width are increased by 1 m, the area increases by 16 m^2 and the volume by 64 m^3. Find the dimensions of the room if its length and width are in the ratio of 7:3.

   ℓ = 10.5, w = 4.5

9. ABCD is a square with sides 2 cm long. Two circular arcs are drawn from B to D, one with A as centre and the other with C as centre. What is the area of the region enclosed by the two circular arcs?

   2π - 4 or 2.3

10. A swimming pool 4 m by 6 m is inside a rectangular garden such that each corner of the pool touches a different side of the garden. What is the largest possible area of the garden?

    50 m^2

11. A cabin is built on a flat, circular island. The ocean is 12 km due north and 6 km due south, and the centre of the island is 7 km away. What is the radius of the island?

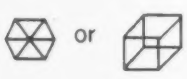
    11 km

12. Which fits better, a round peg in a square hole or a square peg in a round hole?

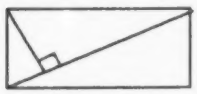
    Round peg in square hole

13. An isosceles triangle has sides of length p-q, p-q and 2q (p>q). Show that its area A is given by $A = q(p^2 - 2pq)^{\frac{1}{2}}$.

14. How can 12 matches be arranged to make 6 regions of equal area?



15. Draw lines to separate a rectangle into 3 similar triangles.

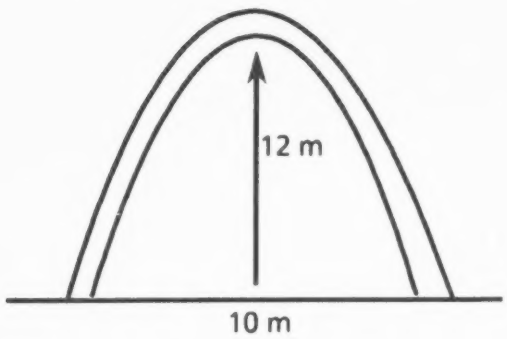


(One solution)

16. The number of square centimetres in the surface area of a sphere is equal to the number of cubic centimetres in the volume. What is its radius?

3 cm

17. A bridge is in the shape of a parabolic arch, as shown. It has a maximum height of 12 m and a width of 10 m. Can a truck carrying oil well equipment to Fort McMurray with height 8 m and width 4 m pass under this bridge?



Yes

18. An eccentric artist claims that her best canvasses have the same area and perimeter (that is, the number of units of area equals the number of units of perimeter). If the unit measures are integers, what are the dimensions of her canvasses?

4 units x 4 units or
3 units x 6 units

19. If a triangle with sides of 6 units, 8 units and 10 units is revolved about one of its sides, a solid figure can be produced. Each side creates a different figure. Which of the three possible figures has the greatest volume? The least volume?

6 unit side produces the greatest volume, 128π units^3, while the revolving around the 10 unit side produces the least volume, 76.8π units^3

20. How many circular pipes with an inside diameter of two centimetres would be needed to carry the same amount of water as a pipe with an inside diameter of thirty centimetres?

225

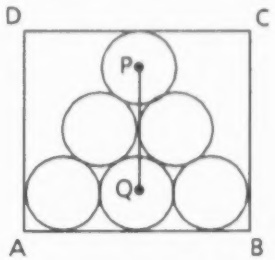
21. An equilateral triangle and a regular hexagon have equal perimeters. What is the ratio of their areas?

T:H = 2:3

22. On the sides of a right triangle, construct three similar figures. How do the sum of the areas on the sides compare to the area on the hypotenuse?

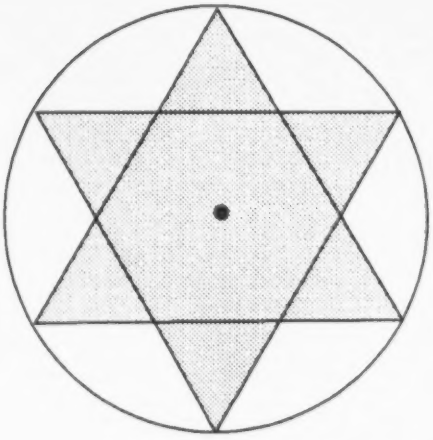
Equal

23. If the area of each circle in the figure below is π 4 and the distance between the centre of the circles P and Q is $\sqrt{3}$, what is the area of rectangle ABCD not covered by the circles? (The circles are congruent, and are tangent to each other and to the sides of the rectangle.)



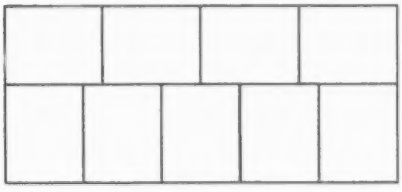
$3\sqrt{3} + 3 - 3\pi/2$

24. The design below is a regular inscribed six-pointed star. If each side of the star is 20 cm long, what is the total area of the unshaded portion?



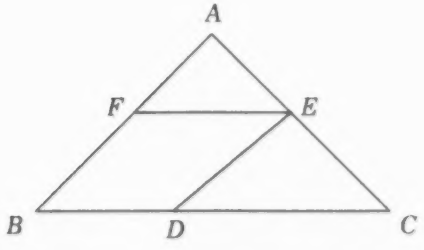
1200 ($\pi - \sqrt{3}$) cm
or 1691 cm

25. Nine congruent rectangles are placed as shown to form a large rectangle whose area is 720 cm^2. Find the perimeter of this figure.



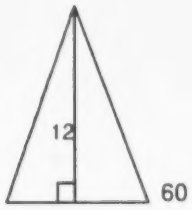
P = 116

26. Rhombus $BDEF$ is inscribed in ΔABC as shown. If $AB = 10$ and $BC = 15$, find DE



DE = 6

27. If the perimeter of an isosceles triangle is 36 and the altitude to the base is 12, what is the area?



p = 36

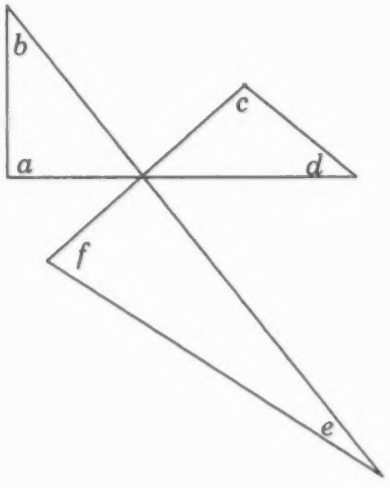
28. In a can of tennis balls that is exactly 3 balls high, which is greater:

    a. the volume of the balls or the volume of the air around them?

    b. the distance around the lid of the can or the height of the can?

a. same
b. around lid

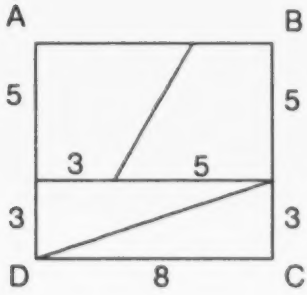
29. What is the sum of angles a, b, c, d, e, and f?

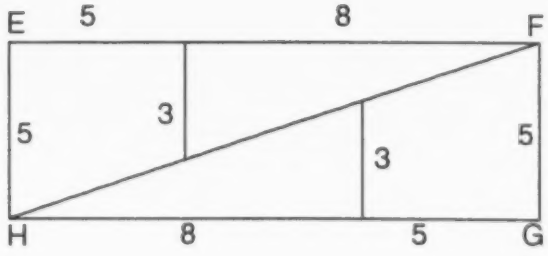


360°

TRIGONOMETRY

1. A square ABCD is cut as shown. It is reassembled to form a rectangle EFGH. The area of ABCD is 64 square units, while EFGH is 65 square units. Explain the discrepancy.





The diagram is deceiving

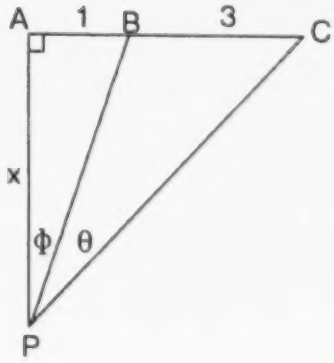
2. A famous outdoorsman was canoeing upstream with his faithful dog. The dog shifted positions and unwittingly pushed the picnic basket out of the canoe. The man did not realize that his lunch was gone until 10 min later. At that time he turned around and began to paddle after the wayward repast. He caught it 1 km from where it had dropped out of the canoe. What is the rate of the current in the stream?

3 km/h

3. In the figure, AB = 1 cm, BC = 3 cm, AP = x cm, $\angle PAC = 90°$, and $\angle BPC = \theta$. By writing $\angle BPA = \Phi$ and using the identity

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

show that $\tan\theta = \dfrac{3x}{x^2+4}$



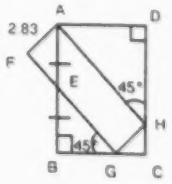
4. Before going on a spare, Michael looked at his watch. After the spare, he noted that the hour hand and the minute hand had exchanged places. How long was the spare?

Various answers around 55 min

5. One angle of a triangle is 60° and the length of the 3 sides of the triangle are an arithmetic sequence. If the remaining angles are A and B, show that

$\sin A + \sin B = \sqrt{3}$.

6. Find the area of rectangle ABCD and rectangle AFGH from the information given in the diagram.



ABCD = 48
AFGH = 24.03

7. Simplify:
$$\frac{\sin A}{\frac{\tan A}{\frac{\sec A}{\frac{\operatorname{ctn} A}{\frac{\cos A}{\csc A}}}}}$$

$\sin^2 A$

8. How many degrees are in the acute angle formed by the hands of a clock at 1:15?

52.5°

9. In ΔABC, $a = 7$, $b=8$, and $c=9$. Find the length of the altitude of the triangle to side c.

$$\frac{8\sqrt{5}}{3}$$

10. In ΔABC $m\angle A = 60°$, $m\angle B = 45°$, and $m\angle C = 75°$, with $AC = 8$. What are the exact lengths of the other two sides?

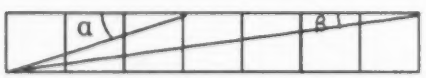
$AB = 4 + 4\sqrt{3}$
$BC = 4\sqrt{6}$

11. Car A starts travelling east along a road. At the same time, from the same point, car B starts travelling north at a speed 15 km/h faster than that of car A. After one hour and twenty minutes, the cars are 100 km apart. At what speeds are they travelling?

45 and 60

12. Given the rectangle consisting of 7 congruent squares, prove that the sum of angle α and β satisfies the following:

$26.5° < \alpha + \beta < 26.6°$



STATISTICS AND PROBABILITY

1. Three different natural numbers are such that the median is 15, the mean is 16 and the range is 7. What are the three numbers?

13, 15, 20

2. Three numbers have a mean of zero and a standard deviation of two. Three other numbers have a mean of zero and a standard deviation of one. Taking the six numbers together, find

   a. their mean
   b. their standard deviation

a. 0
b. $\frac{\sqrt{5}}{2}$

3. The average age of 7 children and a dog is 13 a. Without the dog the average age of the children is 14 a. Find the age of the dog.

6

4. From a hat containing 20 consecutively numbered tickets, two tickets are drawn at random without replacement. If the smaller number is subtracted from the larger, what is the probability that the difference of the two numbers is ten or less?

290/380

5. Four dice have the numbers 7, 8, 9, 10, 11, 12 on each of them. If they are all thrown, what is the probability of obtaining a sum of 40?

125/1296

6. There are three wheels each with five different numbers from 1, 2, 3,... through 15. No two wheels (may) have the same number on them. The wheels are spun and on any wheel any of the five numbers is equally likely to come up. One wheel wins over another when it has a higher number showing. Find numbers for wheels A, B, and C such that the probability that wheel A wins over wheel B is greater than 1/2; the probability that wheel B wins over wheel C is greater than 1/2; the probability that wheel C wins over wheel A is greater than 1/2. Also calculate the three probabilities. (Problem-Solving and Mathematics Competition, Senior, p. 29.)

   Answer: There are many solutions. One such solution is

   A: 15, 10, 8, 6, 1; P(A wins over B) = 13/25
   B: 14, 12, 7, 5, 3; P(B wins over C) = 14/25
   C: 13, 11, 9, 4, 2; P(C wins over A) = 13/25

7. The arithmetic mean (average) of five numbers is 2. If the smallest of the five numbers is deleted from the set, the average of the remaining numbers is 4. What is the smallest number in the original set?

–6

8. Two people agree to meet for lunch between 12:00 noon and 1:00 p.m. If each agrees to wait 15 min for the other, what is the probability that the two will *not* lunch together?

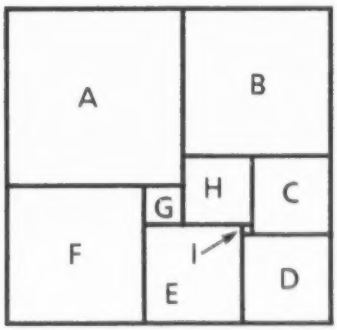
9/16

GENERAL PROBLEMS

1. Manuel, Curt, and John are married to Michelle, Tanya, and Stacey, but not in this order. Each weekend, they play tennis doubles but the wives and husbands are not partners. Tanya and John were partners and Curt was with Manuel's wife. If Curt and Stacey were partners, who is married to whom?

Manuel and Stacey
Curt and Tanya
John and Michelle

2. The following diagram shows an arrangement of squares that appeared on the cover of the November 1958 issue of *Scientific American*. If the area of square C is 64 square units and the area of square D is 81 square units, what are the areas of the other seven squares?



A is 324 units^2
B is 225 units^2
C is 64 units^2
D is 81 units^2
E is 100 units^2
F is 196 units^2
G is 16 units^2
H is 49 units^2
I is 1 unit^2

3. How can you bring up from the river exactly six litres of water when you have only two containers, a four litre pail and a nine litre pail to measure with?

Various answers

4. A pencil, eraser, and notebook cost $1.00. A notebook costs more than 2 pencils, 3 pencils cost more than 4 erasers, and 3 erasers cost more than a notebook. What does each cost?

Notebook 55¢
Pencils 26¢
Erasers19¢

5. One of five brothers had broken a window. John said, "It was Henry or Thomas". Henry said, "Neither Ernest nor I did it". Thomas said, "You are both lying". David said, "No, one of them is speaking the truth, but not the other". Ernest said, "No David, that is not true". Three of the brothers always tell the truth, but the other two cannot be relied on. Who broke the window?

Thomas

6. In the Calgary Herald, the sports writing staff picked the winners for the first weekend of play in the Canadian Football League's football season. The picks are as follows:

Sports Writer A	Sp. Wt. B	Sp. Wt. C
Edmonton	Ottawa	Calgary
Montreal	B.C.	Edmonton
Calgary	Edmonton	Winnipeg
Saskatchewan	Montreal	Ottawa

None of the sports writers picked Toronto or Hamilton.

Who plays on the first scheduled weekend?

B.C. - Calgary
Edmonton - Toronto or Hamilton
Saskatchewan - Ottawa
Winnipeg - Montreal

7. Imagine that you have 12 cannonballs, all but one of equal weight. You are not told whether the odd cannonball is heavier or lighter than the others, only that it is different. You are given a balance scale and told to try to find the odd cannonball. How can you do this in only 3 weighings?

8. At one time, movies were inexpensive. Adults paid 10¢, teens paid 4¢, and children 12 and under entered at ten for a penny. If a hundred people attended a show for a dollar, how many of each -- adults, teens and children -- were there?

2 Adults
18 Teens
80 Children

9. The political commentator on Station CNOR summarized an election result as follows: "A Tail Wagger Party victory of 1729 votes in the last election has been turned into a Rescuer Party victory of 1654 in this election. The Rescuer Party candidate received 38% of the votes cost. The Tail Wagger Party candidate finished in second place. The Best Friend Party received only 14% of the votes cast and was beaten for third place by the Sheepherder Party by 50 votes." If there were only four candidates in the election, determine the number of votes each candidate received. (Super Problems, p. 57.)

Rescuer Party - 15238 votes
Tail Wagger Party - 13584 votes
Best Friend Party - 5614 votes
Sheepherder Party - 5664 votes

10. Fred and Frank run a race. Fred runs half the time and walks half the time. Frank runs half the distance and walks half the distance. If they both walk and run at the same rate, who wins?

Fred

11. At an international competition among twenty-four students, three languages were spoken (French, Spanish, and English). Eight of the students who spoke English did not speak Spanish. Eight of the students who spoke Spanish did not speak English. Of the eight students who spoke French, three spoke only French. Six spoke only English, and only one student at the competition spoke all three languages. How many students spoke Spanish and English but not French?

12. On the planet Urp most years have 180 days (18 months of 10 days each). Once every 7 years, however, an Urp year contains 181 d (the 3rd month contains 11 days). Each Urp week is 5 d long. The days of the week, in order of occurrence, are Oneday, Twoday, Threeday, Fourday, and Fiveday. Duorps was born on a Fourday, on the 1st day of the 4th month in a 181-day year. On what day of the week was Duorps' 25th birthday?

Twoday

13. Sam and Suzie are brother and sister. Sam has as many brothers as sisters. Suzie has twice as many brothers as sisters. How many girls are in their family?

3

14. Two explorers plan to travel in a straight line and return by the same route. Each can carry a 12-day supply of food and travel 30 km/d. If they are not required to return at the same time, what is the farthest distance that can be explored? (Both must return!)

240 km

15. If 1 zilch is equal to 13 milches and 1 milch is equal to 23 pilches, would you accept 8000 pilches for 26 zilches?

Yes

16. The total number of angles in two regular polygons is thirteen, and the total number of diagonals is twenty-five. How many angles are in each polygon?

5 and 8

4

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APPENDIX A

Contacts for Contests and Leagues in Canada

1. The Canadian Mathematics Competition

   Cayley Contest - Grade 10
   Fermat Contest - Grade 11
   Euclid Contest - Grade 12

   Faculty of Mathematics
   University of Waterloo
   Waterloo, Ontario
   N2L 3G1

   In Alberta contact
   Dr. Robert Woodrow
   Department of Mathematics and Statistics
   University of Calgary
   2500 University Drive, N.W.
   Calgary, Alberta
   T2N 1N4

2. Alberta High School Mathematics Prize Exam

   Chairman of Alberta High School Mathematics Prize Exam
   Mathematics Department
   University of Alberta
   Edmonton, Alberta
   T6G 2E2

   Chairman of Alberta High School Mathematics Prize Exam
   Mathematics Department
   University of Calgary
   2500 University Drive, N.W.
   Calgary, Alberta
   T2N 1N4

3. Canadian Mathematics Olympiad

   Canadian Mathematical Society
   577 King Edward Avenue
   Ottawa, Ontario
   K1N 6N5
   Phone (613) 564-2223

4. Canadian National Mathematics League

   Mathematics Department
   University of Windsor
   Windsor, Ontario
   N9B 3P4
   (affiliated with Mathematics Leagues Inc.)

APPENDIX B

Professional Organizations for Mathematics Teachers

The Mathematics Council of the Alberta Teachers' Association
11010 - 142 Street
Edmonton, Alberta
T5N 2R1

The National Council of Teachers of Mathematics
1906 Association Drive
Reston, VA, USA
22091

School Science and Math Association Inc.
126 Life Science Building
Bowling Green State University
Bowling Green, OH, USA
43403

Alberta Teachers' Association, Computer Council
11010 - 142 Street
Edmonton, Alberta
T5N 2R2